Prof. Girardi	Math 142	Spring 2006	04.29.06	Final Exam

MARK BOX		
PROBLEM	POINTS	
1 a – y	25	
2 a – o	15	
3	3	
4 a – c	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
15	5	
16	5	
POSSIBLE	108	

NAME: Solutions

please check the box of your section

☐ **Section 001 (MW 9:05 am)**

or

☐ **Section 002 (MW 10:10 am)**

INSTRUCTIONS:

(1) To receive credit you must:
   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
   (b) if a line/box is provided, then:
   — show you work BELOW the line/box
   — put your answer on/in the line/box
   (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): S 7.1 - 7.4, 7.6, 7.7, 8.1 - 8.5, 8.8, 10.1-10.10, 11.1 - 11.3. .

1. **Fill in the blanks (each worth 1 point).**

1a. $\int \frac{du}{u} = $ ln $|u| + C$

1b. If a is a constant and $a > 0$ but $a \neq 1$, then $\int a^u\, du = $ $\frac{a^u}{\ln a}$ $+C$

1c. $\int \cos u\, du = $ sin u $+C$

1d. $\int \sec^2 u\, du = $ tan u $+C$

1e. $\int \sec u \tan u\, du = $ sec u $+C$

1f. $\int \sin u\, du = $ - cos u $+C$

1g. $\int \csc^2 u\, du = $ - cot u $+C$

1h. $\int \csc u \cot u\, du = $ - csc u $+C$

1i. $\int \tan u\, du = $ -ln| cos u| $+C$

1j. $\int \cot u\, du = $ ln | sin u | $+C$

1k. $\int \sec u\, du = $ ln | sec u + tan u | $+C$

1l. $\int \csc u\, du = $ - ln |csc u + cot u| $+C$

1m. If a is a contant and $a > 0$ then $\int \frac{1}{\sqrt{a^2-u^2}}\, du = $ $\sin^{-1} \frac{u}{a}$ $+C$

1n. If a is a contant and $a > 0$ then $\int \frac{1}{a^2+u^2}\, du = $ $\frac{1}{a} \tan^{-1} \frac{u}{a}$ $+C$

1o. If a is a contant and $a > 0$ then $\int \frac{1}{u\sqrt{u^2-a^2}}\, du = $ $\frac{1}{a} \sec^{-1} \frac{|u|}{a}$ $+C$

1p. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do long division

1q. Integration by parts formula: $\int u\, dv = $ $uv - \int v\, du$

1r. Trig substitution: (recall that the *integrand* is the function you are integrating) if the integrand involves a^2-u^2, then one makes the substitution $u = $ $a \sin \theta$

1s. Trig substitution: if the integrand involves a^2+u^2, then one makes the substitution $u = $ $a \tan \theta$

1t. Trig substitution: if the integrand involves u^2-a^2, then one makes the substitution $u = $ $a \sec \theta$

1u. trig formula ... your answer should involve trig functions of θ, and not of 2θ: $\sin(2\theta) = $ $2 \sin\theta \cos\theta$.

1v. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$.

1w. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$.

1x. trig formula ... since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween tangent (i.e., tan) and secant (i.e., sec) is $\sec^2\theta - \tan^2\theta = 1$.

1y. $\arctan(\frac{-1}{\sqrt{3}}) = $ $-\pi/6$ **RADIANS**. (your answer should be an angle)

$\int \frac{\cos u}{\sin u}$ $u = \sin u$ $du = \cos$ $\int \frac{1}{u} du$ $\ln|\sin u|$

$\tan = \frac{\sin}{\cos}$

$\frac{\pi}{6}: \frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ $-\frac{\pi}{6}$ $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$

$\cancel{\frac{5\pi}{6}: \frac{1}{2} \cdot -\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}}$ $-\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

(cos, sin)

2. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

2a. Sequences Let $-\infty < r < \infty$. (Fill-in-the blanks with *exists* or *does not exist, i.e. DNE*)

- $\to 0$ • If $|r| < 1$, then $\lim_{n\to\infty} r^n$ exists
- $\to \infty$ • If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist
- $\to 1$ • If $r = 1$, then $\lim_{n\to\infty} r^n$ exists
- $\to$ osc • If $r = -1$, then $\lim_{n\to\infty} r^n$ does not exist

2b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

2c. p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

2d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

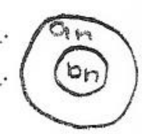
Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a decreasing function
- f is a continuous function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

2e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.



2f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges .

2g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

2h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$ decreasing
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

2i. n^{th}**-term test** for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges .

2j. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is <u>absolutely convergent</u> if and only if $\sum |a_n|$ converges
- $\sum a_n$ is <u>conditionally convergent</u> if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is <u>divergent</u> if and only if $\sum a_n$ diverges

2k. Consider a **function** $y = f(x)$ where $f\colon [1, \infty) \to \mathbb{R}$.
Next consider the corresponding **sequence** $\{a_n\}_{n=1}^{\infty}$ where $a_n \stackrel{\text{def.}}{=} f(n)$.

- If the limit of the **function** $y = f(x)$ as $x \to \infty$ is L,

  then the limit of the corresponding **sequence** $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$ is L.
- If $\lim_{n\to\infty} a_n = L$, is it necessarily true that $\lim_{x\to\infty} f(x) = L$? Circle: Yes or $\boxed{\text{No}}$

for **2l – 2o**

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = p_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = p_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

2l. In open form (i.e., with … and without a $\sum$-sign)

$$p_N(x) = f^0(x_0) + f'(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^N(x_0)(x-x_0)^N}{N!}$$

2m. In closed form (i.e., with a $\sum$-sign and without …)

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

2n. In closed form (i.e., with a $\sum$-sign and without …)

$$p_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

2o. We know that $f(x) = p_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)(x-x_0)^{N+1}}{(N+1)!}$$ for some c between $\boxed{x}$ and $\boxed{x_0}$.

3. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=17}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge?
Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$?

if NO $\Rightarrow$ $\sum a_n$ is divergent

(from "Does $\sum |a_n|$ converge?") if YES $\Downarrow$

$\sum a_n$ is absolutely convergent

(from "Does $\lim_{n\to\infty} |a_n| = 0$?") if YES $\Downarrow$

Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is conditionally convergent

$4x+1=x$
$4x-x+1=0$
$3x+1=0$ $3x=-1$
$x=-1/3$

4. Let R be the region in the $x-y$-plane that is enclosed by

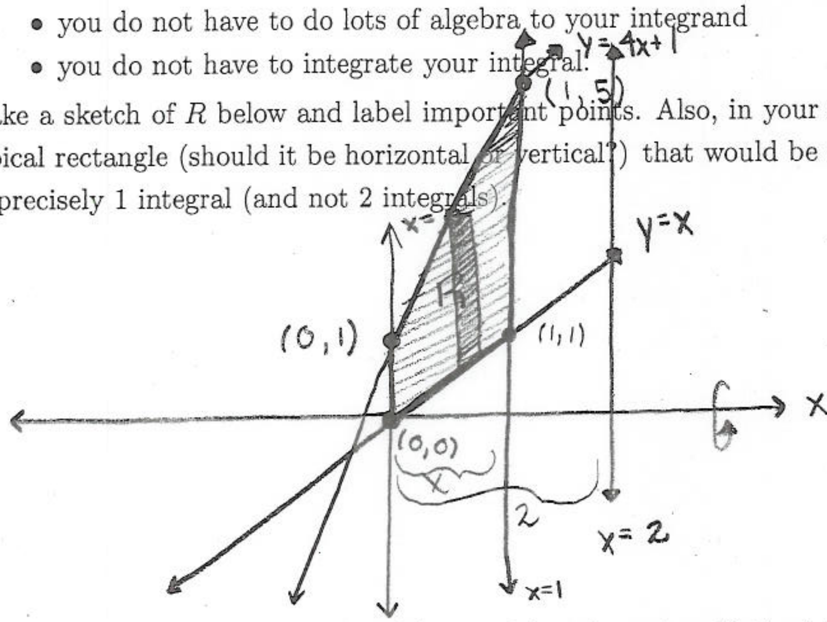
$$y = x \quad \text{and} \quad y = 4x+1 \quad \text{and} \quad x = 0 \quad \text{and} \quad x = 1\,.$$

x	y
0	1
1	5
2	9

In each of problems **4b and 4c**:

- R will be revolved around some line to create a solid of revolution
- using either the disk, washer, or shell method, express the volume V of the resulting solid of revolution as **one integral** (and NOT as 2 or more integrals).
- In the space provided **below** each problem, make some *good enough sketch* (does not have to be too fancy) to indicate (i.e., help justify) your thinking/reseasoning behind your solution
- you do not have to do lots of algebra to your integrand
- you do not have to integrate your integral.

4a. Make a sketch of R below and label important points. Also, in your sketch of R below, draw in a typical rectangle (should it be horizontal or vertical?) that would be used to express the area of R as precisely 1 integral (and not 2 integrals).



4b. The volume V of the solid obtained by revolving the region R about the x-axis is

$$\boxed{V = \pi \int_{x=0}^{x=1} \left((4x+1)^2 - (x)^2 \right) dx}$$

Hint $(r_{\text{big}}^2 - r_{\text{little}}^2) \neq (r_{\text{big}} - r_{\text{little}})^2$

continued next page ⇒

4c. The volume V of the solid obtained by revolving the region R about the line $x = 2$ is

$$V = 2\pi \int_{x=0}^{x=1} \left((4x+1-x)(2-x) \right) dx$$

Hint: the line $x = 2$ is a vertical line (it is NOT a horizontal line).

2π (average height) (radius) (thickness)

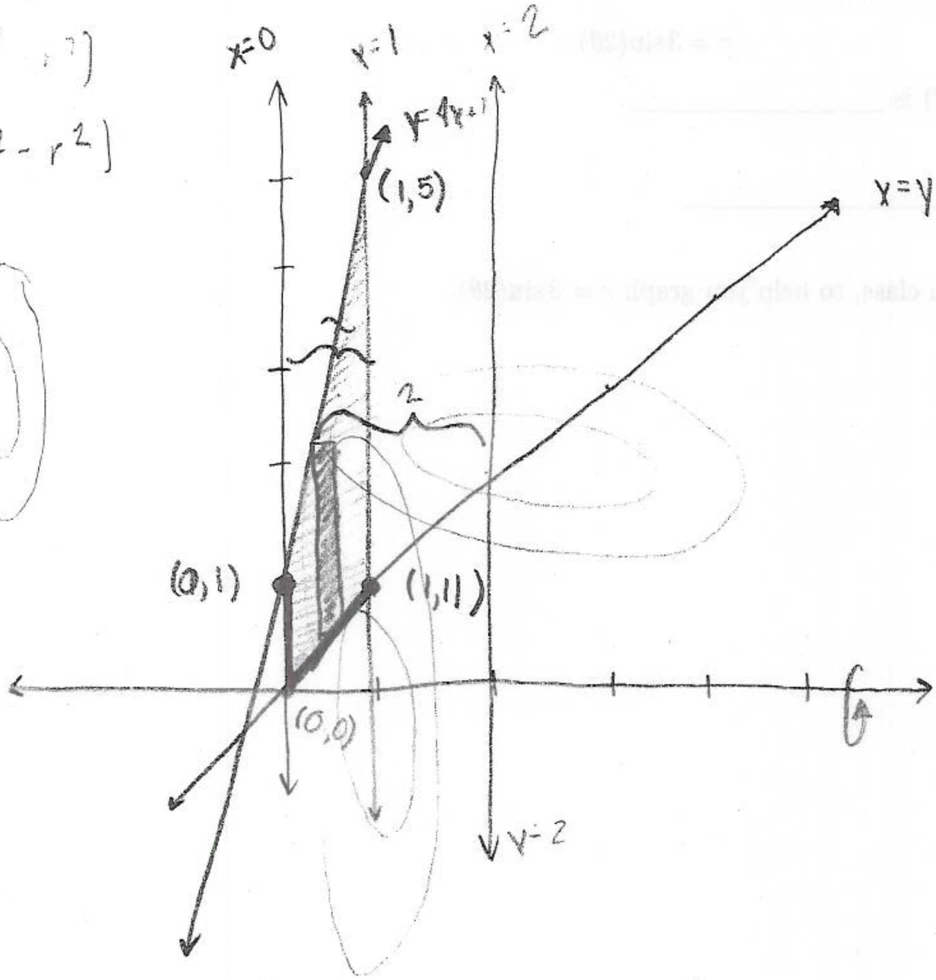
$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$(4x+1-x) \qquad (2-x) \qquad \Delta x$$

$\pi r^2 \cdot \pi r^2$

$r(r^2 \cdot r^2)$

$r(r^2 - r^2)$



$$V = \pi \int_{x=0}^{x=1} \left((4x+1)^2 - (x)^2 \right) dx$$

$$V = 2\pi \int_{x=0}^{x=1} (\text{ave radius})(\text{height})(\text{thickness})$$

$$2\pi \int_{x=0}^{x=1} (4x-1 - x)(2-x)\, dx$$

$$\sec^2 x - \tan^2 x = 1$$
$$\tan^2 x = \sec^2 x - 1$$

5. $\displaystyle\int \sec^3 x \tan^3 x \, dx = \frac{1}{5}\sec^5 x + {}^{-}\frac{1}{3}\sec^3 x \qquad +C$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

$$\int \sec^3 x \tan^3 x \, dx =$$

$$\int (\sec^2 x \tan^2 x (\sec x \tan x))\, dx =$$

$$\int (\sec^2 x (\sec^2 x - 1)(\sec x \tan x))\, dx =$$

$$\int ((\sec^4 x - \sec^2 x)(\sec x \tan x))\, dx =$$

$$\boxed{\begin{array}{l} u = \sec x \\ du = \sec x \tan x \, dx \end{array}}$$

$$\int (u^4 - u^2)\, du =$$

$$\frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$\frac{1}{5}\sec^5 x + {}^{-}\frac{1}{3}\sec^3 x + C$$

$\sec^2\theta = 1 + \tan^2\theta$

6. $$\boxed{\int \frac{dx}{(4+x^2)^2} = \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{32}\left(2\left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right)\right) \qquad +C}$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

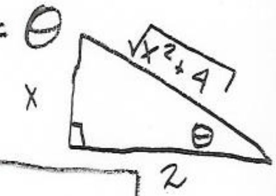
Hint: useful might be problems: 1r – 1w.

$\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2$

$$\int \frac{dx}{(2^2+x^2)^2}$$

$\tan^{-1}\frac{x}{2} = \theta$



$u^2 + a^2$

$x = a\tan\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$x^2 = 4\tan^2\theta$

$\tan\theta = \frac{x}{2}$

$(4+x^2) = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$

$(4+x^2)^2 = (4\sec^2\theta)^2 = 16\sec^4\theta$

$\sec = \frac{1}{\cos}$

$$\int \frac{dx}{(4+x^2)^2} = \int \frac{2\cancel{\sec^2\theta}\, d\theta}{\cancel{16}_8 \sec^{\cancel{4}\,2}\theta} = \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta$$

$\frac{1}{\sec^2\theta} = \frac{1}{\frac{1}{\cos^2\theta}} = \cos^2\theta$

$$\frac{1}{8}\int \cos^2\theta\, d\theta = \frac{1}{8}\int\left(\frac{1+\cos(2\theta)}{2}\right)d\theta = \frac{1}{8}\cdot\frac{1}{2}\int(1+\cos(2\theta))d\theta$$

$$\frac{1}{16}\int 1\, d\theta + \frac{1}{16}\int\cos(2\theta)\, d\theta = \frac{1}{16}\theta + \frac{1}{16}\cdot\frac{1}{2}\int \cos u\, du$$

$\boxed{\begin{array}{l} u = 2\theta \\ du = 2d\theta \\ \frac{1}{2}du = d\theta \end{array}}$

$$= \frac{1}{16}\theta + \frac{1}{32}\sin(2\theta) + \phi$$

$$= \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{32}\left(2\left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right)\right) + \phi$$

$Ax^3+Ax+Bx^2+B$

7. $$\int \frac{x^3+x^2+2x+1}{x^4+2x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + \tan^{-1}x - \frac{1}{2}(x^2+1)^{-1} + C$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

Hint: $x^4+2x^2+1=(x^2+1)^2$, which is a irreducible quadatic squared.

$$\int \frac{x^3+x^2+2x+1}{(x^2+1)^2} \Rightarrow \frac{Ax+B}{(x^2+1)}^{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} = \frac{Ax^3+Ax+Bx^2+B+Cx+D}{(x^2+1)^2}$$

x^3: $\boxed{A=1}$

x^2: $\boxed{B=1}$

x : $A+C=2$

constant : $B+D=1$

$1+C=2$

$\boxed{C=1}$

$1+D=1$

$\boxed{D=0}$

$u=x^2+1$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$u=x^2+1$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$$\int \frac{Ax+B}{(x^2+1)} + \int \frac{Cx+D}{(x^2+1)^2} = \int \frac{x+1}{x^2+1}dx + \int \frac{x}{(x^2+1)^2}dx$$

$$= \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx + \int \frac{x}{(x^2+1)^2}dx =$$

$$= \frac{1}{2}\int \frac{1}{u}\,du + \int \frac{1}{x^2+1}dx + \frac{1}{2}\int \frac{1}{u^2}\,du \quad u^{-2}$$

$$\frac{1}{2}\ln|x^2+1| + \int \frac{1}{x^2+1}dx + \frac{1}{2}\cdot -1u^{-1} + C$$

$$\frac{1}{2}\ln|x^2+1| + \tan^{-1}x + -\frac{1}{2}(x^2+1)^{-1} + C$$

8. $\boxed{\int x^3 e^{x^2}\,dx = \frac{1}{2}e^{x^2}x^2 - \frac{1}{2}e^{x^2} \qquad +C}$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

Hint: one lesson from parts was to look for a dv that is easy to integrate and thus get the v.

$$\int x^2 x e^{x^2}\,dx$$

$$\boxed{\begin{array}{ll} u = x^2 & dv = xe^{x^2}dx \\ du = 2x\,dx & v = \frac{1}{2}e^{x^2} \end{array}}$$

$$\boxed{\begin{array}{ll} \int xe^{x^2}dx & u = x^2 \\ \frac{1}{2}\int e^u du & du = 2x\,dx \\ \frac{1}{2}e^u + C & \frac{1}{2}du = x\,dx \\ \frac{1}{2}e^{x^2} & \end{array}}$$

$$\int x^3 e^{x^2}dx = \frac{1}{2}e^{x^2}x^2 - \int 2\cdot\frac{1}{2}xe^{x^2}dx$$

$$= \frac{1}{2}e^{x^2}x^2 - \int xe^{x^2}dx$$

$$= \frac{1}{2}e^{x^2}x^2 - \frac{1}{2}\int e^u du$$

$$\boxed{\begin{array}{l} u = x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{array}}$$

$$= \frac{1}{2}e^{x^2}x^2 - \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2}x^2 - \frac{1}{2}e^{x^2} + C$$

9. $\boxed{\int \sin(\ln x)\,dx \ = \ \dfrac{x\sin(\ln x) - x\cos(\ln x)}{2} \qquad +C}$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

Hint: one lesson from parts was that if the integrand is easy to differentiate but hard to integrate, then let u be in integrand. Another lesson from parts was the bring to the other side idea.

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\cancel{\tfrac{1}{x}}\cancel{x}\,dx$$

$$\boxed{\begin{array}{ll} u = \sin(\ln x) & dv = dx \\ du = \cos(\ln x)\frac{1}{x}\,dx & v = x \end{array}} \qquad \boxed{\begin{array}{ll} u = \cos(\ln x) & dv = dx \\ du = -\sin(\ln x)\frac{1}{x}\,dx & v = x \end{array}}$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \cancel{x}\sin(\ln x)\cancel{\tfrac{1}{x}}\,dx$$

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx$$

$$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)\,dx = \frac{x\sin(\ln x) - x\cos(\ln x)}{2}$$

10. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=17}^{\infty} (-1)^n \frac{1}{\sqrt{n^2-1}}$$

- [ ] ~~absolutely convergent~~
- [x] conditionally convergent
- [ ] divergent

abs conv? $\sum |a_n| = \sum \frac{1}{\sqrt{n^2-1}} = \sum \frac{1}{(n^2-1)^{1/2}}$ n big ≈ $b_n = \frac{1}{n}$

C.T

$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{1}{(n^2-1)^{1/2}} \cdot n = \lim_{n\to\infty} \frac{n}{(n^2-1)^{1/2}}$

$\lim_{n\to\infty} \frac{n'}{n' + \text{junk}} = 1$ $0 < 1 < \infty$ $\therefore$ $\sum |a_n|$ + $\sum b_n$ do the same thing

$\sum b_n = \sum \frac{1}{n}$ diverges (harmonic series or p test where p=1)

$\therefore$ the given series is not abs conv. ($\Rightarrow$ diverges)

cond conv? A.S.T

$\lim_{n\to\infty} |a_n| \overset{?}{=} 0$; $\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{1}{(n^2-1)^{1/2}} = \frac{1}{(\infty)^{1/2}} = 0$ ✓

$f(x) = |a_n|$ + $f(x)$ is decreasing

$f(x) = (n^2-1)^{-1/2}$

$f'(x) = -\frac{1}{2}(n^2-1)^{-3/2}(2n) < 0$ ✓

$\therefore$ The given series is cond convergent due to the A.S.T

$2n+2\cdot 1$

$2n+1$

11. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=8}^{\infty} (-1)^n \frac{n!}{(2n-1)!}$$

- [x] absolutely convergent
- [ ] ~~conditionally convergent~~
- [ ] ~~divergent~~

But before you get started .... let

$$a_n = \frac{n!}{(2n-1)!}.$$

Then $a_{n+1} = \boxed{\dfrac{(n+1)!}{(2n+1)!}}$

Next, simplify $\frac{a_{n+1}}{a_n}$ so that it has NO factorial sign (that is a ! sign) in it.

$$\frac{a_{n+1}}{a_n} = \boxed{\frac{n+1}{(2n+1)(2n)}}$$

Ok, now you should be ready to finish off the problem and check the correct box above.

abs conv?

$$a_n = \frac{n!}{(2n-1)!} \quad a_{n+1} = \frac{(n+1)!}{(2(n+1)-1)!} \quad a_{n+1} = \frac{(n+1)!}{(2n+2-1)!} \quad a_{n+1} = \frac{(n+1)!}{(2n+1)!}$$

Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+1)!} \cdot \frac{(2n-1)!}{n!}$$

$$\lim_{n\to\infty} \frac{(n+1)\cancel{(n)!}}{(2n+1)(2n)\cancel{(2n-1)!}} \cdot \frac{\cancel{(2n-1)!}}{\cancel{n!}} = \lim_{n\to\infty} \frac{n+1}{(2n+1)(2n)}$$

$$\lim_{n\to\infty} \frac{n+1}{4n^2+2n} = \frac{\infty}{\infty} \overset{LH}{=} \lim_{n\to\infty} \frac{1}{8n+2} = \frac{1}{\infty} = 0$$

$L = 1$ inc
$L<1$ con
$L>1$ div

$\therefore$ The given series ~~series~~ is abs convergent b/c $\sum |a_n|$ converges by the ratio test

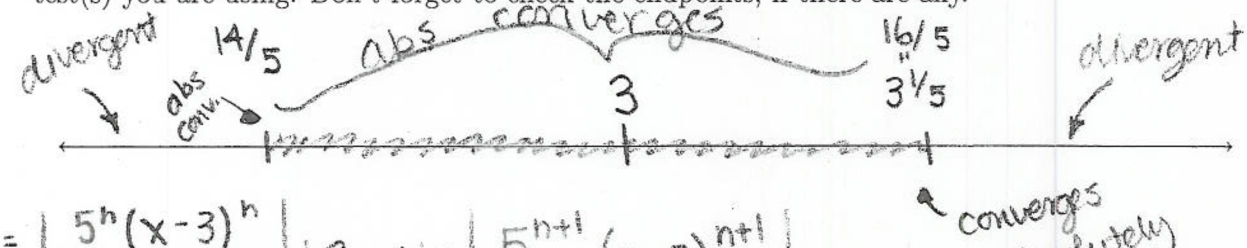
12. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{5^n\,(x-3)^n}{n^2}$$

Hint: $\left|\frac{5^{n+1}(x-3)^{n+1}}{(n+1)^2}\ \frac{n^2}{5^n(x-3)^n}\right| \;=\; \frac{5^{n+1}}{5^n}\,\frac{|x-3|^{n+1}}{|x-3|^n}\,\frac{n^2}{(n+1)^2} \;=\; 5\ |x-3|\,\left(\frac{n}{n+1}\right)^2.$

The center is $x_0 =$ 3 and the radius of convergence is $R =$ 1/5.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$|a_n| = \left|\frac{5^n(x-3)^n}{n^2}\right| \quad |a_{n+1}| = \left|\frac{5^{n+1}(x-3)^{n+1}}{(n+1)^2}\right|$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\cancel{5^n}\cdot 5\,\cancel{(x-3)^n}(x-3)}{n^2+2n+1} \cdot \frac{n^2}{\cancel{5^n}\,\cancel{(x-3)^n}}\right|$

$-3| = \frac{5n^2}{n^2+2n+1}$

tio Test: ✓

$x-3| \lim_{n\to\infty} \frac{5n^2}{n^2+2n+1} = 5|x-3| < 1$

$|x-3| < 1/5$

$|x - +3| < 1/5$

center Radius

See attached for endpoints

~~endpoints: $x = 14/5$ $\sum_{n=1}^{\infty} \frac{5^n(14/5-3)^n}{2^n} = \sum_{n=1}^{\infty} \frac{5^n(-\frac{1}{5})^n}{2^n} = \sum_{n=1}^{\infty} \frac{5^n(-1)^n(\frac{1}{5})}{2^n}$ abs conv? $|a_n| = \frac{5^n}{2^n}$~~

~~root test $\frac{1}{5}\lim_{n\to\infty}\left(\frac{5^n}{2^n}\right)^{1/n} = \frac{1}{5}\lim_{x\to\infty}\frac{5}{2} = \frac{5}{2}\cdot\frac{1}{2} = 5/4$ L=1 inc, L<1 con, L>1 div, not abs conv~~

~~nd conv? $\lim_{n\to\infty} a_n = \frac{1}{5}\lim_{n\to\infty}\frac{5^n}{2^n} = \frac{1}{5}\cdot\frac{5^\infty}{2^\infty} = \infty$ divergent~~

4]

endpoints :

$= 14/5$

$$\sum_{n=1}^{\infty} \frac{5^n(14/5-3)^n}{n^2} \qquad \sum_{n=1}^{\infty} \frac{5^n(-\frac{1}{5})^n}{n^2} = \sum_{n=1}^{\infty} \frac{5^n(-1)^n(1/5)^n}{n^2}$$

abs conv? $\sum |a_n| \quad \sum \frac{5^n(\frac{1}{5})^n}{n^2} = \sum \frac{1^{2n}}{n^2} = \sum \frac{1}{n^2}$ p-test $p=2$ $p>1$ converges absolutely

$= 16/5$ $\qquad 16/5 - \frac{15}{5} = 1/5$

$$\sum_{n=1}^{\infty} \frac{5^n(16/5-3)^n}{n^2} = \sum_{n=1}^{\infty} \frac{5^n(1/5)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1^{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\therefore$ converges absolutely

$\sum \frac{1}{n}$ divg

$\sum \frac{(-1)^n}{n}$ cc.

$$\frac{4n^3+3n}{3n^2+3} \quad \frac{12n^2+6n}{6n} = \frac{24n+6}{6} = \infty$$

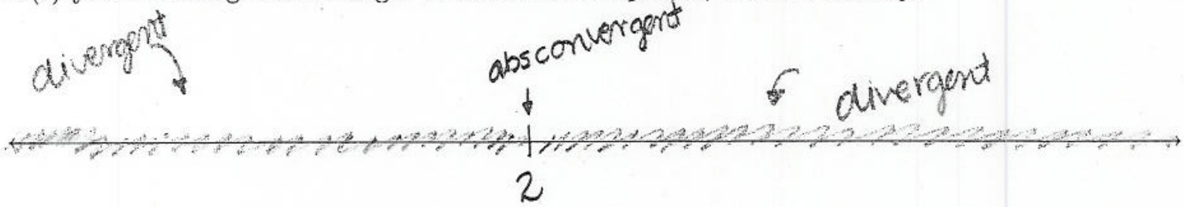
13. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^3}\,(x-2)^n$$

Hint: do the same kind of calculation as done in the hint for the previous problem.

The center is $x_0 =$ 2 and the radius of convergence is $R =$ 0.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



Ratio Test: $a_{n+1} = \dfrac{(n+1)!}{(n+1)^3}\,(x-2)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \quad \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-2)^{n+1}}{(n+1)^3}\cdot\frac{n^3}{n!\,(x-2)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}\,\cancel{(x-2)^n}(x-2)}{(n+1)^3}\cdot\frac{n^3}{\cancel{n!}\,\cancel{(x-2)^n}}\right| = |x-2|\lim_{n\to\infty}\frac{(n+1)(n^3)}{(n+1)^3}$$

$$|x-2|\lim_{n\to\infty}\frac{n^4+n^3}{(n+1)^3} = |x-2|\lim_{n\to\infty}\frac{n^4+n^3}{n^3+3n+1} = \frac{\infty}{\infty}$$

$$|x-2|\lim_{n\to\infty}\frac{1+\frac{1}{n}}{\frac{1}{n}+\frac{3}{n^3}+\frac{1}{n^4}} = |x-2|\cdot\underset{\text{divergent}}{\infty} = \infty < 1$$

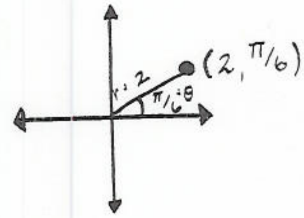
$$|x-2|\cdot\infty \not< 1$$

$$\infty \not< 1$$

$$|x-2| < \frac{1}{\infty}\; 0$$

center radius

14. Consider the point, in **polar coordinates**,



$$P = (r\,,\,\theta) = \left(2, \frac{\pi}{6}\right).$$

In **cartesian coordinates**, the point P is given by

$$P = (x, y) = (\ \sqrt{3}\ ,\ 1\ ).$$

Below graph, and CLEARLY label, the following points.

$$P = \left(2, \frac{\pi}{6}\right)$$
$$Q = \left(-2, \frac{\pi}{6}\right)$$
$$R = \left(2, -\frac{\pi}{6}\right)$$
$$S = \left(-2, -\frac{\pi}{6}\right).$$

$r = 2$

$\theta = \frac{\pi}{6}$

$x = r\cos\theta$

$x = 2\cos\left(\frac{\pi}{6}\right)$

$x = \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}$

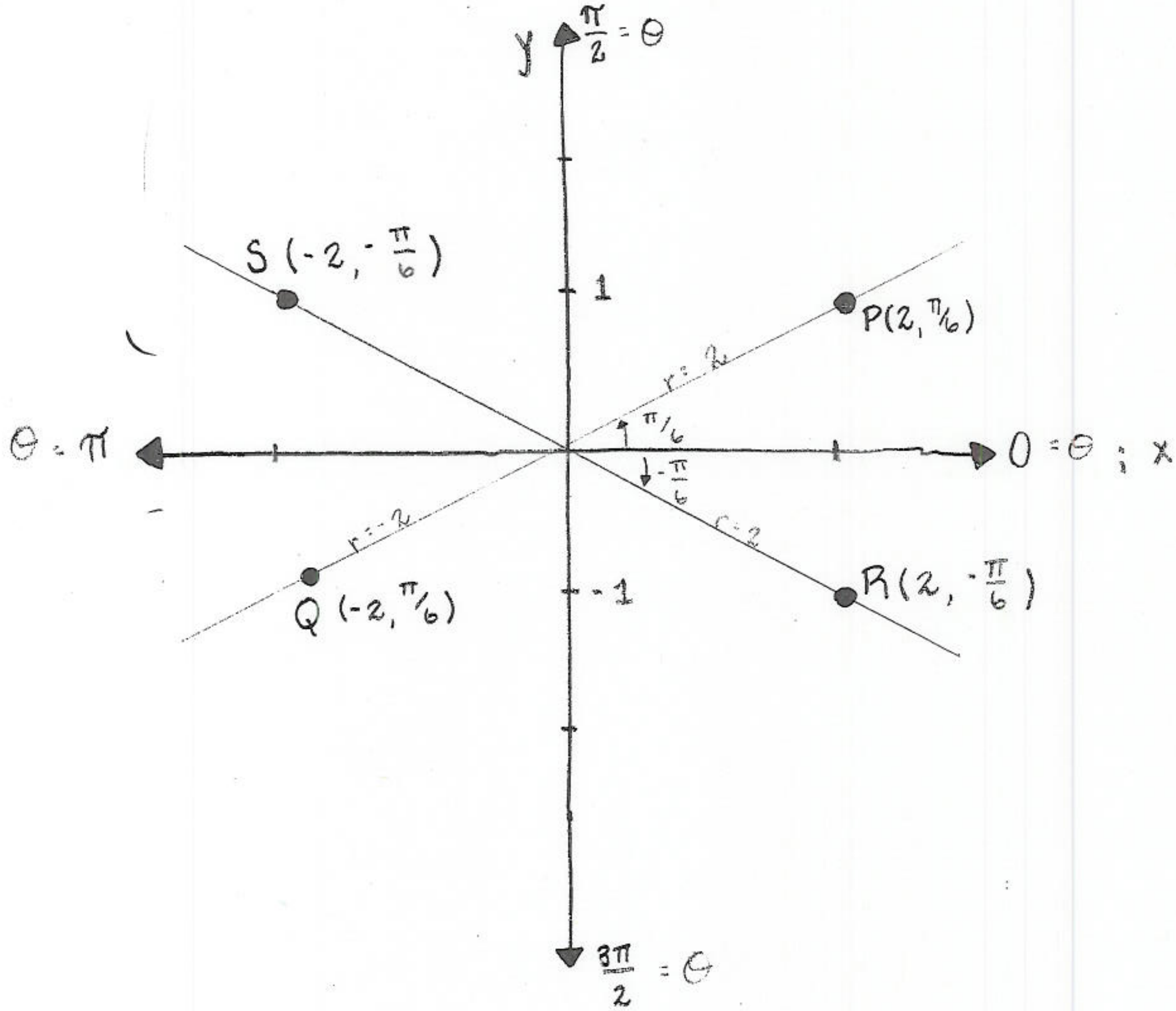
$\boxed{x = \sqrt{3}}$

$y = r\sin\theta$

$y = 2\sin\left(\frac{\pi}{6}\right)$

$y = \cancel{2}\cdot\frac{1}{\cancel{2}}$

$\boxed{y = 1}$



15. Consider the curve in polar coordinate

$$r = 3\sin(2\theta)\ .$$

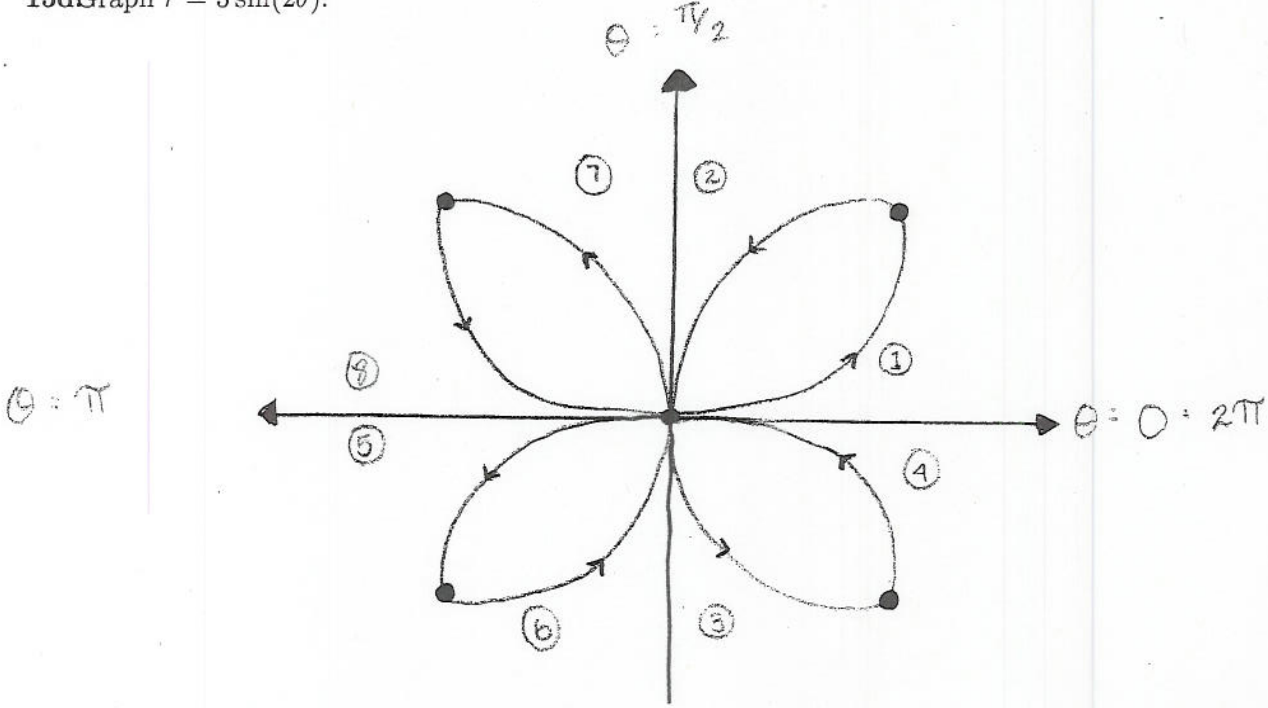
15a. The period of $r = 3\sin(2\theta)$ is $\frac{2\pi}{2} = \pi$.

15b. $\frac{\text{the period of } r = 3\sin(2\theta)}{4} = \frac{\pi}{4}$

15c. Make a chart, as we did in class, to help you graph $r = 3\sin(2\theta)$.

	θ	2θ	$\sin(2\theta)$	$r = 3\sin(2\theta)$
1	$0 \to \frac{\pi}{4}$	$0 \to \pi/2$	$0 \to 1$	$0 \to 3$
2	$\frac{\pi}{4} \to \frac{\pi}{2}$	$\frac{\pi}{2} \to \pi$	$1 \to 0$	$3 \to 0$
3	$\frac{\pi}{2} \to \frac{3\pi}{4}$	$\pi \to 3\pi/2$	$0 \to -1$	$0 \to -3$
4	$\frac{3\pi}{4} \to \pi$	$\frac{3\pi}{2} \to 2\pi$	$-1 \to 0$	$-3 \to 0$
5	$\pi \to \frac{5\pi}{4}$	$2\pi \to$	$0 \to 1$	$0 \to 3$
6	$\frac{5\pi}{4} \to \frac{3\pi}{2}$	"just go around the unit circle again"	$1 \to 0$	$3 \to 0$
7	$\frac{3\pi}{2} \to \frac{7\pi}{4}$		$0 \to -1$	$0 \to -3$
8	$\frac{7\pi}{4} \to 2\pi$		$-1 \to 0$	$-3 \to 0$

15d Graph $r = 3\sin(2\theta)$.



16. Express the area enclosed by $r = 3\sin(2\theta)$ as an integral with respect to θ (ok ... with respect to θ means a $d\theta$ in there).
(You do not have to evaluate this integral.)

area = $\frac{1}{2} \cdot 4 \int_{\theta=0}^{\theta=\pi/2} [3\sin(2\theta)]^2 \, d\theta$

$$A = \frac{1}{2} \int_{\theta=a}^{\theta=\beta} [f(\theta)]^2 \, d\theta$$ where $r = f(\theta)$

$$A = \frac{1}{2} \cdot 4 \int_{\theta=0}^{\theta=\frac{\pi}{2}} [3\sin(2\theta)]^2 \, d\theta$$

due to symmetry