MARK BOX		
PROBLEM	POINTS	
1 a – o	15	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	13	
%	100	

NAME: Solutions

please check the box of your section

☐ **Section 001 (MW 9:05 am)**

or

☐ **Section 002 (MW 10:10 am)**

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) if a line/box is provided, then:
    — show you work BELOW the line/box
    — put your answer on/in the line/box
  (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): the whole of Chapter 10: Sections 10.1 - 10.10 .

Problem Inspiration:

1. From class handouts. You were warned.
2. homework problem § 10.1 # 15 and Mo's Friday homework # 2
3. Serious Series Problems # 4
4. Mo's Friday homework # 2
5. homework problem § 10.8 # 35
6. Example from class.
7. Mo's Friday homework # 2
8. homework problem § 10.9 # 3

1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1a. Sequences Let $-\infty < r < \infty$. (Fill-in-the blanks with *exists* or *does not exist, i.e. DNE*)

- 0 • If $|r| < 1$, then $\lim_{n\to\infty} r^n$ exists
- ∞ • If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist
- 1 • If $r = 1$, then $\lim_{n\to\infty} r^n$ exists
- osc. • If $r =$ -1, then $\lim_{n\to\infty} r^n$ does not exist

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r| < 1$
- diverges if and only if $|r| \geq 1$

1c. ***p*-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if $p > 1$
- diverges if and only if $p \leq 1$

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f \colon [1, \infty) \to \mathbb{R}$ be so that

- $a_n = f(\ n\ )$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a decreasing or nonincreasing function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

$\frac{2n}{3n}$

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

a_n b_n

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges .

1g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If $\rho < 1$ then $\sum a_n$ converges.
- If $\rho > 1$ then $\sum a_n$ diverges.
- If $\rho = 1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If

- $a_n > a_{n+1}$ for each $n \in \mathbb{N}$ decreasing
- $\lim_{n\to\infty} a_n = 0$

then $\sum(-1)^n a_n$ converges

1i. n^{th}**-term test** for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges .

1j. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1k. Consider a **function** $y = f(x)$ where $f\colon [1, \infty) \to \mathbb{R}$.
Next consider the corresponding **sequence** $\{a_n\}_{n=1}^{\infty}$ where $a_n \overset{\text{def.}}{=} f(n)$.

- If the limit of the **function** $y = f(x)$ as $x \to \infty$ is L,

  then the limit of the corresponding **sequence** $\{a_n\}_{n=1}^{\infty}$ as $n \to \infty$ is L.
- If $\lim_{n\to\infty} a_n = L$, is it necessarily true that $\lim_{x\to\infty} f(x) = L$? Circle: **Yes** or $\boxed{\textbf{No}}$

for **1l – 1o**

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.
Let $y = p_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.
Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.
Let $y = p_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

1l. In open form (i.e., with … and without a $\sum$-sign)

$$p_N(x) = f^{0}(x_0) + f^{(1)}(x_0)(x-x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N$$

1m. In closed form (i.e., with a $\sum$-sign and without …)

$$p_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1n. In closed form (i.e., with a $\sum$-sign and without …)

$$p_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

1o. We know that $f(x) = p_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{(N+1)}$$ for some c between $\boxed{x}$ and $\boxed{x_0}$.

2. For the following **SEQUENCES**:

- if the limit exists, find it
- if the limit does not exist, then say that it DNE.

Put your ANSWER IN the box and show your WORK BELOW the box.

2a. $\lim_{n\to\infty} \frac{(3n+1)(5n+2)}{17n^2} = \frac{15}{17}$

$\lim_{n\to\infty} \frac{15n^2+6n+5n+2}{17n^2} = \lim_{n\to\infty} \frac{15n^{2}+11n+2}{17n^{2}}$

2b. $\lim_{n\to\infty} (-1)^n \frac{(3n+1)(5n+2)}{17n^2} = \text{DNE}$

$\lim_{n\to\infty} \frac{(-1)^n(15n^2+11n+2)}{17n^2} = \left(\begin{array}{c}\text{divergence}\\ \text{osc. btw } 1 \text{ \& } -1\end{array}\right) \cdot \frac{15}{17} =$ divergence b/c osc.

2c. $\lim_{n\to\infty} (1.00000017)^n = \text{DNE}$

$\lim_{n\to\infty} r^n$ and $|r| > 1$ then does not exist

$1.00000017 > 1$ so

3. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^8+1}}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

abs. conv?

$$|a_n| = \frac{1}{(n^8+1)^{1/2}} \overset{n \text{ big}}{\approx} \frac{1}{(n^8)^{1/2}} = \frac{1}{n^4} \equiv b_n$$

LCT

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(n^8+1)^{1/2}} \cdot \frac{n^4}{1} = \lim_{n\to\infty} \frac{(n^8)^{1/2}}{(n^8+1)^{1/2}}$$

$$= \lim_{n\to\infty} \left(\frac{n^8}{n^8+1}\right)^{1/2} = \left(\frac{1}{1}\right)^{1/2} = 1 \equiv L$$

$0 < L < \infty \implies \Sigma a_n$ & Σb_n do same thing

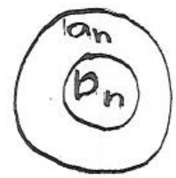
$\Sigma b_n = \Sigma \frac{1}{n^4}$ $\quad p = 4 > 1$ p-series conv.

So $\Sigma |a_n|$ converges

4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=8}^{\infty} (-1)^n \frac{1}{(\ln n)^3}$$

- ☐ ~~absolutely convergent~~
- ☑ conditionally convergent
- ☐ divergent



Hint: For any $0 < q < \infty$, if n is big enough then $\ln n < n^q$ and so $\underbrace{\frac{1}{(n^q)^3}}_{b_n} < \underbrace{\frac{1}{(\ln n)^3}}_{a_n}$.
Hint: the integral test is NOT helpful.

• abs convergent? $|a_n| = \frac{1}{(\ln n)^3} \overset{n \text{ big}}{>} b_n = \frac{1}{(n^q)^3}$

↑ need b_n to diverge
where q is ?
I'll say $q = 1/4$

guess: $\frac{1}{(\ln n)^3} \approx \frac{1}{n^{3/4}}$ converge p-test

CT: $|a_n| = \frac{1}{(\ln n)^3} > b_n = \frac{1}{(n^{1/4})^3} = \frac{1}{n^{3/4}}$

$\sum b_n = \sum \frac{1}{n^{3/4}}$ diverges; p-test
$p = 3/4 \quad p < 1$

$\therefore$ Since $\sum b_n$ diverges & $b_n < a_n$ then $\sum a_n$ also diverges by the CT so the given series is not abs convergent

• conditionally convergent?

1. $\lim_{n \to \infty} |a_n|$ must equal 0

$\lim_{n \to \infty} \frac{1}{(\ln n)^3} = \frac{1}{(\infty)^3} = \frac{1}{\infty} = 0$ ✓

2. $f(x) = |a_n|$ and $f(x)$ must be decreasing

$f(x) = \frac{1}{(\ln n)^3} \quad f(x) = (\ln n)^{-3}$

$f'(x) = -3(\ln n)^{-4}\left(\frac{1}{n}\right)$

$f'(x) = \frac{-3}{(\ln n)^4 n} < 0$: dec ✓

$\therefore$ The given serie is conditionally convergent by the A.S.T.

$2(n+1)+1$
$2n+2+1$
$2n+3$

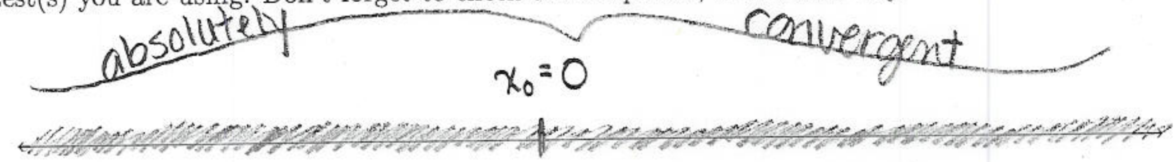
5. Consider the formal power series

$$\sum_{n=1}^{\infty} (-1)^n \, \frac{x^{2n+1}}{(2n+1)!}$$

Hint: $\left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \, \frac{(2n+1)!}{x^{2n+1}} \right| = \frac{|x|^{2n+3}}{|x|^{2n+1}} \, \frac{(2n+1)!}{(2n+3)!} = \frac{|x|^2}{1} \, \frac{\cancel{(2n+1)!}}{\cancel{(2n+1)!}\,(2n+2)\,(2n+3)}$.

The center is $x_0 =$ 0 and the radius of convergence is $R =$ ∞ .

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$$\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad |a_n| = \frac{x^{2n+1}}{(2n+1)!} \qquad |a_{n+1}| = \frac{x^{2n+3}}{(2n+3)!}$$

ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{2n}} \cdot \cancel{x^{3}}\, x^2}{(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{(2n+1)!}}{\cancel{x^{2n}} \cdot \cancel{x}} \right|$$

$$|x|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = \cancel{\frac{1}{\infty}} = 0 \cdot |x|^2 < 1 \quad \text{to converge}$$

$0 < 1$ always; $\mathbb{R}$

Interval of convergence: $(-\infty, \infty)$

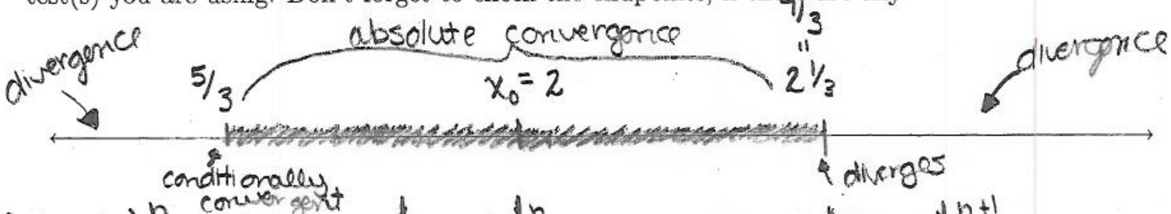
6. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(3x-6)^n}{n}$$

Hint: $(3x-6)^n = [3(x-2)]^n = 3^n\,(x-2)^n$.

The center is $x_0 =$ 2 and the radius of convergence is $R =$ 1/3 .

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$$\sum_{n=1}^{\infty} \frac{(3x-6)^n}{n} \qquad |a_n| = \frac{|3x-6|^n}{n} \qquad |a_{n+1}| = \frac{|3x-6|^{n+1}}{n+1}$$

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{(3x-6)^n}(3x-6)}{n+1} \cdot \frac{n}{\cancel{(3x-6)^n}} \right| =$$

$$|3x-6| \lim_{n\to\infty} \frac{n}{n+1} \text{ LH} = |3x-6| \lim_{n\to\infty} \frac{1}{1} = 1 \cdot |3x-6| < 1 \text{ to converge}$$

$$|3x-6| < 1$$
$$|3(x-2)| < 1$$
$$|x-2| < 1/3$$
$$|x - +2| < 1/3$$

center radius

endpoints

$x = 5/3$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{(\cancel{3}(5/\cancel{3}) - 6)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

abs conv? $\sum |a_n| \quad \Sigma \frac{1}{n}$ diverges; harmonic series or p=1
∴ not abs converge

cond conv? ① $\lim_{n\to\infty} |a_n| \quad \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$ ✓

② $f(x) = |a_n| \quad f(x) = n^{-1} \quad f'(x) = -n^{-2} < 0$ dec ✓ ∴ The given series is conditionally convergent at $x = 5/3$ by the A.S.T

$x = 7/3$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{(\cancel{3}(7/\cancel{3}) - 6)^n}{n} \quad \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$; diverges harmonic series or p=1 ∴ The given series diverges at $x = 7/3$ by the p-test

7. Using the fact that

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \qquad \text{when} \qquad |r| < 1\,, \qquad (*)$$

find a power series expansion of

$$\frac{x}{2+32x^4}$$

and state when it is valid. Simplify your answer so that your power series has the form $\sum_{n=0}^{\infty} c_n x^{\text{some power}}$ for some constants c_n.

$$\boxed{\frac{x}{2+32x^4} = \sum_{n=0}^{\infty} \frac{(-16)^n}{2} x^{(4n+1)} \text{ or } \sum_{n=0}^{\infty} \frac{(-1)^n 16^n}{2} x^{4n+1} \quad \text{when } |x| < \frac{1}{2}}$$

Hint: $32 = 2(16)$ and $16 = 2^4$.

$$g(x) = \frac{x}{2+32x^4}$$

$$= x\left[\frac{1}{2+32x^4}\right]$$

$$= \frac{x}{2}\left[\frac{1}{1+16x^4}\right]$$

$$= \frac{x}{2}\left[\frac{1}{1-\,-(16)x^4}\right]$$

every x replaced by $-16x^4$ in the original;

$$|\cancel{-}16x^4| < 1$$

$$|16x^4| < 1$$

$$|x^4| < \frac{1}{16}$$

$$|x| < \frac{1}{\sqrt[4]{16}}$$

$$|x| < \frac{1}{2}$$

$$g(x) = \sum_{n=0}^{\infty} \frac{x}{2}(-16x^4)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^1(-16)^n(x^4)^n}{2}$$

$$= \sum_{n=0}^{\infty} \frac{x^1(-16)^n(x^{4n})}{2}$$

$$\sum_{n=0}^{\infty} \frac{x^{4n+1}(-16)^n}{2}$$

$$\sum_{n=0}^{\infty} \frac{(-16)^n}{2} x^{4n+1} \qquad (-1\cdot 16)^n$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 16^n}{2} x^{4n+1}}$$

8. In this problem, you may **NOT** use a known Taylor series; instead, you must compute requested items by hand. Let

$$f(x) = e^{-x} \qquad \text{and} \qquad x_0 = 17\ .$$

We will follow the notation from this exam on **PAGE 3** problems **1l - 1o**, so here

- $y = p_\infty(x)$ is the Taylor series of $y = e^{-x}$ about $x_0 = 17$
- $y = p_N(x)$ is the N^{th}-order Taylor polynomial of $y = e^{-x}$ about $x_0 = 17$
- $y = R_N(x)$ is the N^{th}-order Taylor remainder of $y = e^{-x}$ about $x_0 = 17$.

You may use, without showing, that

$$f^{(n)}(x) = (\text{-}1)^n\, e^{-x}$$

for $n = 0, 1, 2, 3, 4, \ldots$.

8a. In closed form (i.e., with a $\sum$-sign and without ...)

$$p_\infty(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n e^{-17}}{n!} (x-17)^n}$$

Your answer should NOT have the symbol $f^{(n)}$ in it. $(-1)^{N+1}(-1)^{N+1} = (-1)^{2N+2} = +1$

8b. We know that $f(16) = p_N(16) + R_N(16)$. Taylor's BIG Theorem tells us that, for $x = 16$,

$$R_N(16) = \boxed{\frac{(-1)^{N+1} e^{-c}}{(N+1)!} (16-17)^{N+1} \overset{\text{or}}{=} \frac{e^{-c}}{(N+1)!}}$$ for some c between $\boxed{16}$ and $\boxed{17}$.

Your answer should have a N and a c in it but it should NOT have a x nor x_0 in it.

8c. Find an upper bound for $|R_N(16)|$.

$$|R_N(16)| \le \boxed{\frac{1}{(N+1)!} \frac{1}{e^{16}}}$$

Your answer should have an N in it but should NOT have a c nor x nor x_0 in it.

$$|R_N(16)| = \frac{1}{(N+1)!} \frac{1}{e^c} \le \frac{1}{(N+1)!} \frac{1}{e^{16}}$$

$$16 < c < 17 \Rightarrow e^{16} < e^c < e^{17} \Rightarrow \frac{1}{e^{17}} < \frac{1}{e^c} < \frac{1}{e^{16}}$$

8d. Show that $f(16) = p_\infty(16)$.

$$|R_N(16)| \le \frac{1}{e^{16}} \frac{1}{(N+1)!} \xrightarrow{N\to\infty} 0$$

$$\Rightarrow \lim_{N\to\infty} R_N(16) = 0$$

$$\Rightarrow f(16) = p_\infty(16).$$