Prof. Girardi | Math 142 | Spring 2006 | 03.02.06 | Exam 2

MARK BOX		
PROBLEM	POINTS	
1 a–j	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
%	100	

NAME: ____________________________

please check the box of your section

☐ **Section 001 (MW 9:05 am)**

or

☐ **Section 002 (MW 10:10 am)**

INSTRUCTIONS:

(1) To receive credit you must:
   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
   (b) if a line/box is provided, then:
      — show you work BELOW the line/box
      — put your answer on/in the line/box
   (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): Sections 8.1, 8.2, 8.3, 8.4, 8.5, 8.7, 8.8. .

Problem Inspiration:

1. You were warned.
2. An example from class.
3. A typical example from that section.
4. homework problem § 8.3 # 37
5. Handout of 100 integrals # 26
6. Handout of 100 integrals # 50
7. An example from class.
8. homework problem § 8.2 # 1
9. homework problem § 8.8 # ~~[illegible]~~ 57 a
10. homework problem § 8.2 # 58a and the last page of the Handout of 100 integrals.

1. Fill in the blanks (each worth 1 point).

1a. If a is a constant and $a > 0$ but $a \neq 1$, then

$\int a^u \, du$ = $\frac{a^u}{\ln a}$ $+C$

1b. $\int \tan u \, du$ = $-\ln|\cos u| + C \overset{\text{or}}{=} \ln|\sec u| + C$

1c. If a is a contant and $a > 0$ then

$\int \frac{1}{a^2+u^2} \, du$ = $\frac{1}{a} \tan^{-1} \frac{u}{a} + C \overset{\text{or}}{=} \frac{1}{a} \arctan \frac{u}{a} + C$

1d. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do long division

1e. Integration by parts formula: $\int u \, dv$ = $uv - \int v \, du$

1f. Trig substitution: (recall that the *integrand* is the function you are integrating) if the integrand involves $a^2 - u^2$, then one makes the substitution u = $a \sin\theta$

1g. Trig substitution:
if the integrand involves $a^2 + u^2$, then one makes the substitution u = $a \tan\theta$

1h. Trig substitution:
if the integrand involves $u^2 - a^2$, then one makes the substitution u = $a \sec\theta$

1i. If you numerically approximate $\int_{x=a}^{x=b} f(x)\, dx$ using the Trapezoid Rule T_n, partitioning the interval $[a, b]$ into n subintervals, and if f'' is continuous on $[a, b]$, then

$$\left| T_n - \int_{x=a}^{x=b} f(x)\, dx \right| \leq \frac{(b-a)^3 \, K_2}{12n^2}$$

where

$K_2 = \max_{a \leq z \leq b} |f''(z)| \overset{\text{or}}{=}$ maximum value of $|f''|$ on $[a,b]$

1j. If you numerically approximate $\int_{x=a}^{x=b} f(x)\, dx$ using Simpson's Rule S_{2n}, partitioning the interval $[a, b]$ into $2n$ subintervals, and if $f^{(4)}$ is continuous on $[a, b]$, then

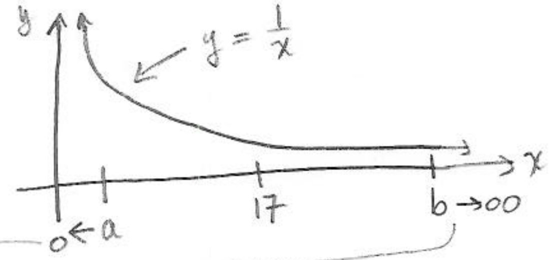
$$\left| S_{2n} - \int_{x=a}^{x=b} f(x)\, dx \right| \leq \frac{(b-a)^5 \, K_4}{180(2n)^4}$$

where

$K_4 = \max_{a \leq z \leq b} |f^{(4)}(z)| \overset{\text{or}}{=}$ maximum value of $|f^{(4)}|$ on $[a,b]$

2. Improper Integral. Use correct notation and verify your reasoning. Convince your reader that you understand what is coming on. A picture is worth 1000 words.

2a. Make a rough sketch of the graph of $y = \frac{1}{x}$ for $0 < x < \infty$.



2b. $\displaystyle\int_{x=0}^{x=17} \frac{1}{x}\,dx =$ diverges or diverges to ∞ or DNE

$$\int_{x=0}^{x=17} \frac{dx}{x} = \lim_{a \to 0^+} \int_a^{17} \frac{dx}{x} = \lim_{a\to 0^+} \ln|x| \Big|_{x=a}^{x=17} = \lim_{a \to 0^+} [\ln 17 - \ln a]$$

$$= \ln 17 - \left[\lim_{a\to 0^+} \ln a\right] = 17 - \text{"}[-\infty]\text{"} = \text{"}\infty\text{"}$$

2c. $\displaystyle\int_{x=17}^{x=\infty} \frac{1}{x}\,dx =$ diverges or diverges to ∞ or DNE

$$\int_{x=17}^{x=\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_{x=17}^{x=b} \frac{dx}{x} = \lim_{b \to \infty} \left[\ln|x| \Big|_{x=17}^{x=b} \right]$$

$$= \lim_{b\to\infty} [\ln b - \ln 17] = \left[\lim_{b\to\infty} \ln b\right] - \ln 17 = \text{"}\infty - 17\text{"} = \text{"}\infty\text{"}$$

2d. $\displaystyle\int_{x=0}^{x=\infty} \frac{1}{x}\,dx =$ diverges or diverges to ∞ or DNE

$$\int_{x=0}^{x=\infty} \frac{dx}{x} = \int_{x=0}^{x=c} \frac{dx}{x} + \int_{x=c}^{x=\infty} \frac{dx}{x}$$ ← for any c where $0 < c < \infty$ & eg $c = 17$

$$= \underbrace{\int_{x=0}^{x=17} \frac{dx}{x}}_{\text{diverge}} + \underbrace{\int_{x=17}^{x=\infty} \frac{dx}{x}}_{\text{diverges}}$$

Since at least one of these two integral diverges, the total integral diverges

3. Let's numerically approximate

$$\int_{x=5}^{x=7} \sqrt{x}\, dx$$

by using **Simpson's Rule** S_{2n}, partitioning the interval $[5,7]$ into $2n = 6$ subintervals.

3a. $S_6 \approx \frac{1}{3}\left(\frac{7-5}{6}\right)\left[1\sqrt{5} + 4\sqrt{\frac{16}{3}} + 2\sqrt{\frac{17}{3}} + 4\sqrt{6} + 2\sqrt{\frac{19}{3}} + 4\sqrt{\frac{20}{3}} + 1\sqrt{7}\right]$

$\Delta x = \frac{\text{length of } [5,7]}{\#\text{ of subintervals}} = \frac{7-5}{6} = \frac{2}{6} = \frac{1}{3}$

Note $\frac{1}{3}\left(\frac{7-5}{6}\right) = \frac{1}{3}\cdot\frac{2}{6} = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$

$a = x_0$	x_1	x_2	x_3	x_4	x_5	$x_6 = b$
$\frac{15}{3} = 5$	$\frac{16}{3}$	$\frac{17}{3}$	$\frac{18}{3} = 6$	$\frac{19}{3}$	$\frac{20}{3}$	$\frac{21}{3} = 7$

3b. Following the notation from problem **1j** of this exam, for problem **3**

$K_4 = \frac{15}{16}\cdot\frac{1}{5^{7/2}}$ or $= \frac{15}{16}\cdot\frac{1}{5^{3+\frac{1}{2}}}$ or $= \frac{15}{16}\cdot\frac{1}{5^3\sqrt{5}}$ or $= \frac{3}{16}\cdot\frac{1}{25\sqrt{5}}$ or etc...

Hint: your answer should be a number and carry out the arithmetic as far as indicated in class.

$f(x) = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} \Rightarrow f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}} \Rightarrow f^{(4)}(x) = \frac{15}{16}x^{-\frac{7}{2}}$.

$K_4 = \max_{a \le z \le b} |f^{(4)}(z)| = \max_{5 \le z \le 7} \left|\frac{15}{16} z^{-\frac{7}{2}}\right| = \frac{15}{16} \max_{5 \le z \le 7} \frac{1}{z^{7/2}}$

and $5 \le z \le 7 \Rightarrow 5^{7/2} \le z^{7/2} \le 7^{7/2} \Rightarrow \frac{1}{7^{7/2}} \le \frac{1}{z^{7/2}} \le \frac{1}{5^{7/2}}$

3c. The error bound for Simpson's Rule is:

$\left|S_6 - \int_{x=5}^{x=7} \sqrt{x}\, dx\right| \le \frac{2^5}{180\,(6)^4}\cdot\frac{15}{16}\cdot\frac{1}{5^{7/2}}$

Hint: your answer should be a number and carry out the arithmetic as far as indicated in class.

$\left|S_6 - \int_{x=5}^{x=7} \sqrt{x}\, dx\right| \le \frac{(7-5)^5}{(180)(6)^4}\cdot\frac{15}{16}\cdot\frac{1}{5^{7/2}}$

w/ calculator just for fun $= \frac{480}{(466560000)(\sqrt{5})} \approx 4.6\,E^{-7}$

4. $\boxed{\displaystyle\int \tan(17x)\ \sec^3(17x)\ dx = \frac{1}{51}\ \sec^3(17x) + C}$

$\boxed{\cancel{\begin{array}{l} s = \tan(17x) \\ ds = 17\sec^2(17x)\,dx \end{array}}}$ $\boxed{\begin{array}{l} t = \sec(17x) \\ dt = 17\sec(17x)\tan(17x)\,dx \end{array}}$

$$\int \tan(17x)\ \sec^3(17x)\ dx$$

$$= \frac{1}{17}\int \sec^2(17x)\ \boxed{17\sec(17x)\tan(17x)\ dx}$$

$$= \frac{1}{17}\int t^2\,dt$$

$$\frac{\overset{2}{17}_{3}}{51}$$

$$= \frac{1}{17}\cdot\frac{t^3}{3} + C$$

$$= \frac{1}{51}\sec^3(17x) + C$$

To check: $D_x\ \frac{1}{51}\sec^3(17x) = \frac{1}{(17)(3)}\cdot(3)\sec^2(17x)\cdot D_x\sec(17x)$

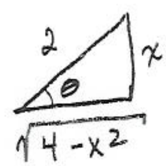
$$= \frac{1}{17}\sec^2(17x)\quad 17\sec(17x)\tan(17x)$$

$$= \tan(17x)\ \sec^3(17x)$$

5. $\boxed{\int \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$

(26) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta \longrightarrow \sin\theta = \frac{x}{2}$



$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\sqrt{1-\sin^2\theta} = 2\cos\theta$

$$\int \frac{x^2\,dx}{\sqrt{4-x^2}} = \int \frac{4\sin^2\theta \;\; 2\cos\theta\, d\theta}{2\cos\theta} = 4\int \sin^2\theta\, d\theta$$

$$= 4\cdot\tfrac{1}{2}\int(1-\cos 2\theta)\,d\theta = 2\int d\theta - 2\,\tfrac{1}{2}\int(\cos(2\theta))\,(2\,d\theta)$$

$$= 2\theta - \sin 2\theta + C = 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C}$$

6. $$\int \frac{x}{x^4+4x^2+8}\,dx =$$

50

$$\int \frac{x}{x^4+4x^2+8}\,dx \underset{\text{square}}{\overset{\text{complete}}{=}} \int \frac{x}{(x^2+2)^2+4}\,dx$$

$$\boxed{\begin{aligned} u &= x^2+2 \\ du &= 2x\,dx \end{aligned}}$$

$$= \frac{1}{2}\int \frac{du}{u^2+2^2} = \frac{1}{2}\cdot\frac{1}{2}\tan^{-1}\frac{u}{2} + c$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x^2+2}{2}\right) + c$$

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or Integrand involves $(x^2+2)^2+4$ ie $u^2+a^2 \Rightarrow u = a\tan\theta$

so
$$\boxed{\begin{aligned} & x^2+2 = 2\tan\theta \longrightarrow \boxed{\tan\theta = \frac{x^2+2}{2}} \\ & 2x\,dx = 2\sec^2\theta\,d\theta \\ & (x^2+2)^2+4 = 4\tan^2\theta+4 = 4(\tan^2\theta+1) = 4\sec^2\theta \end{aligned}}$$

$$\Rightarrow \int \frac{x\,dx}{x^4+4x^2+8} = \int \frac{x\,dx}{(x^2+2)^2+4} = \int \frac{\sec^2\theta\,d\theta}{4\sec^2\theta} = \frac{1}{4}\int d\theta$$

$$= \frac{1}{4}\theta + c = \frac{1}{4}\tan^{-1}\left(\frac{x^2+2}{2}\right) + c.$$

7. $$\boxed{\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx = 2\ln|x| + \frac{1}{x} + \frac{3}{2}\ln|x^2+1| - 2\tan^{-1}x + K}$$

- bigger bottoms? Yes / No [degree num] $= 3 < 4 =$ [degree dem].

- $x^4 + x^2 = x^2(x^2+1)$

$x^2 = (x-0)^2 = (\text{linear term})^2$ ← contribute 2 factors

$b^2 - 4ac = 0^2 - 4(1)(1) < 0 \Rightarrow (x^2+1)^1 = (\text{irred. quad.})^1$ ← 1 factor

$$\frac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} = \frac{Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2}{x^2(x^2+1)}$$

should always be the same

- $\Rightarrow 5x^3 - 3x^2 + 2x - 1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$ — $\boxed{x=0} \Rightarrow \boxed{-1 = B}$

Equate coefficients (short way) ← do on board for them

x^3	:	$5 = A + C$
x^2	:	$-3 = B + D$
x	:	$\boxed{2 = A}$
constant	:	$-1 = B$

$\Rightarrow \boxed{C = 3}$

$\Rightarrow \boxed{D = -2}$

so

$$\int \frac{5x^3-3x^2+2x-1}{x^4+x^2} = \int \left[\frac{2}{x} + \frac{-1}{x^2} + \frac{3x-2}{x^2+1}\right] dx$$

$u = x^2+1$
$du = 2x\,dx$

$$= 2\int \frac{dx}{x} - 1\int x^{-2}\,dx + 3\,\frac{1}{2}\int \underbrace{(x^2+1)^{-1}}_{u^{-1}} \underbrace{(2x\,dx)}_{du} - 2\int \frac{dx}{x^2+1}$$

$$= \boxed{2\ln|x| - \frac{x^{-1}}{-1} + \frac{3}{2}\ln|x^2+1| - 2\tan^{-1}x + K}.$$

8. $\boxed{\int t\, e^{-17t}\, dt = \frac{t\, e^{-17t}}{-17} + \frac{-e^{-17t}}{(17)^2} + C}$

$$\boxed{\begin{array}{ll} u = t & dv = e^{-17t}\, dt \\ du = dt & v = \frac{1}{-17} e^{-17t} \end{array}}$$

Parts

ps: $17^2 = 289.$

$$\int t e^{-17t}\, dt = \frac{t e^{-17t}}{-17} - \frac{1}{-17} \int e^{-17t}\, dt$$

$$= \frac{t e^{-17t}}{-17} + \frac{1}{17} \frac{e^{-17t}}{-17} + C$$

$$\overset{\text{or}}{=} -e^{-17t} \left(\frac{t}{17} + \frac{1}{(17)^2} \right) + C$$

9. LaPlace Transform (from a homework problem)

Consider a function of t, denoted by $y = f(t)$. The **LaPlace Transform** of this function $y = f(t)$ is a (new) function, namely the function

$$y = \mathcal{L}\{f(t)\}\ (s)\ ,$$

which is a function of s for $s > 0$. The formula for the LaPlace Transform of $y = f(t)$ is

$$\mathcal{L}\{f(t)\}\ (s)\ =\ \int_{t=0}^{t=\infty} e^{-st} f(t)\ dt \tag{9}$$

where, in the integral in (9) above, s is treated as a constant.

The LaPlace Transform of the function

$$f(t) = t$$

is the function

$$\boxed{\mathcal{L}\{f(t)\}\ (s)\ =\ \frac{1}{s^2}}$$

for $s > 0$. Hint: your answer should be a function of s and problem **8** of this exam might help.

First, following the ideas from problem 8: $u = t$, $du = dt$, $dv = e^{-st}\,dt$, $v = \frac{-1}{s} e^{-st}$

$$\int t e^{-st}\,dt = -\frac{t}{s} e^{-st} - -\frac{1}{s}\int e^{-st}\,dt = -\frac{t}{s} e^{-st} + \frac{1}{s}\frac{e^{-st}}{-s} + C$$

$$= -e^{-st}\left(\frac{t}{s} + \frac{1}{s^2}\right) + C$$

So

$$\mathcal{L}\{f(t)=t\}(s) = \int_{t=0}^{t=\infty} t e^{-st}\,dt = \lim_{b\to\infty}\int_{t=0}^{t=b} t e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left[-e^{-st}\left(\frac{t}{s}+\frac{1}{s^2}\right)\Big|_{t=0}^{t=b}\right] = \lim_{b\to\infty}\left[-e^{-sb}\left(\frac{b}{s}+\frac{1}{s^2}\right) - -1\left(0+\frac{1}{s^2}\right)\right]$$

$$= \frac{1}{s^2} - \lim_{b\to\infty}\frac{\frac{b}{s}+\frac{1}{s^2}}{e^{sb}} \overset{L'H}{=} \frac{1}{s^2} - \lim_{b\to\infty}\frac{\frac{1}{s}}{s e^{sb}}$$

$$= \frac{1}{s^2} - \frac{1}{s}\cdot\frac{1}{s}\lim_{b\to\infty}\frac{1}{e^{sb}} = \frac{1}{s^2} - \frac{1}{s^2}\cdot 0 = \frac{1}{s^2}$$

10. Let $n \geq 2$. Derive a reduction formula for $\int \sec^n x\, dx$.

$$\boxed{\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx}$$

Show your work, justifying your answer. If you memorized the formula and just write it down and do not show your justification of the formula, you will not receive any points.

Be able to do this Example for Exam 2.

EXAMPLE 6 Find a reduction formula for $\int \sec^n x\, dx$.

Solution The idea is that n is a (large) positive integer, and that we want to express the given integral in terms of a lower power of $\sec x$. The easiest power of $\sec x$ to integrate is $\sec^2 x$, so we proceed as follows.

We don't choose $dv = \sec x\, dx$ because this would introduce a natural logarithm function, a fearsome complication in the second integration.

Let $\quad u = \sec^{n-2} x, \qquad dv = \sec^2 x\, dx.$

Then $\quad du = (n-2) \sec^{n-2} x \tan x\, dx, \qquad v = \tan x.$

This gives

$$\begin{aligned}\int \sec^n x\, dx &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x\, dx \\ &= \sec^{n-2} x \tan x - (n-2) \int (\sec^{n-2} x)(\sec^2 x - 1)\, dx.\end{aligned}$$

Hence

$$\int \sec^n x\, dx = \sec^{n-2} x \tan x - (n-2) \int \sec^n x\, dx + (n-2) \int \sec^{n-2} x\, dx.$$

We solve this equation for the desired integral and find that

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx. \qquad (5)$$