| MARK BOX | | |
|---|---|---|
| PROBLEM | POINTS | |
| 1 a–j | 10 | |
| 2 | 10 | |
| Extra Credit | 1 | |
| 3a | 10 | |
| 3b | 10 | |
| 3c | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| % | 100 | |

**NAME:** Solution Key

**SSN:** ____________________

**please check the box of your section below**

☐ **Section 001 (MW 9:05 pm)**

or

☐ **Section 002 (MW 10:10 pm)**

---

**INSTRUCTIONS:**

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) when applicable put your answer on/in the line/box provided
  (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Anton, Bivens, Davis $8^{\text{th}}$ ed.): Section 7.1 – 7.4, 7.6, 7.7, 8.1 .

---

**Problem Inspiration:**

1. Math 141 HANDOUT
2. homework problem § 7.1 # 15 and 17 , actually problem § 7.1 # 12
3. homework problems from § 7.2 and 7.3 , homework problem § Ch. 7 Review # 6.
4. homework problem § 7.4 # 28a and 30
5. homework problem § 8.1 # 5
6. homework problem § 8.1 # 25
7. homework problem § 8.1 # 19 and a Prof. Girardi quiz
8. the *ice-cream cone* example from class

**1. Fill in the blanks (each worth 1 point).**

**1a.** $\int \frac{du}{u} = \underline{\ln}\ |u| + C$

**1b.** If $a$ is a constant and $a > 0$ but $a \neq 1$, then

$\int a^u\, du = \underline{\frac{a^u}{\ln a}} + C$

**1c.** $\int \sec^2 u\, du = \underline{\tan u} + C$

**1d.** $\int \sec u\ \tan u\ du = \underline{\sec u} + C$

**1e.** $\int \sin u\, du = \underline{-\cos u} + C$

**1f.** $\int \cot u\, du = \underline{\ln |\sin u| + C \ \overset{\text{or}}{=} -\ln |\csc u|} + C$

**1g.** $\int \sec u\, du = \underline{\ln |\sec u + \tan u| + C \ \overset{\text{or}}{=} -\ln|\sec u - \tan u|} + C$

**1h.** If $a$ is a contant and $a > 0$ then

$\int \frac{1}{\sqrt{a^2-u^2}}\, du = \underline{\sin^{-1} \frac{u}{a} + C \ \overset{\text{or}}{=} \ \arcsin \frac{u}{a}} + C$

**1i.** If $a$ is a contant and $a > 0$ then

$\int \frac{1}{a^2+u^2}\, du = \underline{\frac{1}{a} \tan^{-1} \frac{u}{a} + C \ \overset{\text{or}}{=} \ \frac{1}{a} \arctan \frac{u}{a}} + C$

**1j.** The integral of $y = f(x)$ <u>with respect to $x$</u> is denoted by $\int f(x)\, dx$.

The integral of $x = g(y)$ <u>with respect to $y$</u> is denoted by $\underline{\int g(y)\ dy}$.

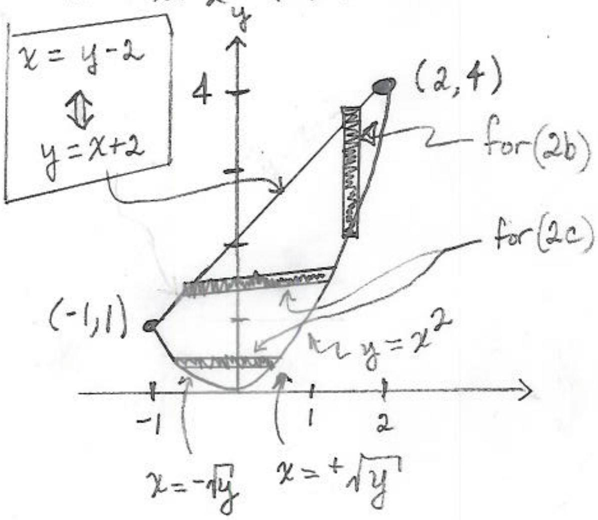
**2.** Let $R$ be the region enclosed by

$$y = x^2 \qquad \text{and} \qquad y = x + 2\,.$$

Let $A$ be the area of the region $R$.

**2a.** The points of intersection of $y = x^2$ and $y = x+2$ are $P = (\underline{-1}, \underline{1})$ and $Q = (\underline{2}, \underline{4})$.
Make a rough sketch of the region $R$, labeling $P$ and $Q$.

$x^2 = x + 2 \iff x^2 - x - 2 = 0 \iff (x-2)(x+1) = 0 \iff x = 2, -1$




**2b.** Express the area $A$ as integral(s) with respect to $x$ (so you want $dx$).
You do NOT have to evaluate the integral(s) nor do lots of algebra.

$$A = \int_{x=-1}^{x=2} \left[ (x+2) - (x^2) \right] dx \ \ \underline{\underline{\text{or}}} \ \int_{x=-1}^{x=2} \left( -x^2 + x + 2 \right) dx$$

**2c.** Express the area $A$ as integral(s) with respect to $y$ (so you want $dy$).
You do NOT have to evaluate the integral(s) nor do lots of algebra.

$$A = \int_{y=0}^{y=1} \left[ (+\sqrt{y}) - (-\sqrt{y}) \right] dy + \int_{y=1}^{y=4} \left[ (+\sqrt{y}) - (y-2) \right] dy$$

$\iff x = \pm\sqrt{y/3}$

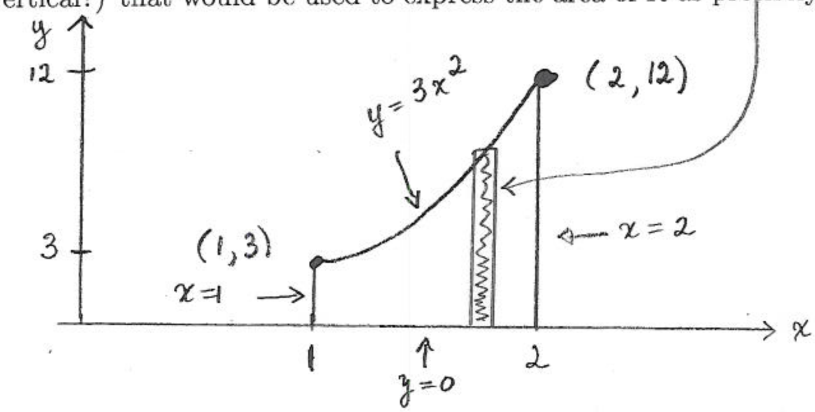
**3.** Sketched below is the region $R$ that is enclosed by

$$y = 3x^2 \quad \text{and} \quad y = 0 \quad \text{and} \quad x = 1 \quad \text{and} \quad x = 2\,.$$

In each of problems **3a, 3b, 3c**:

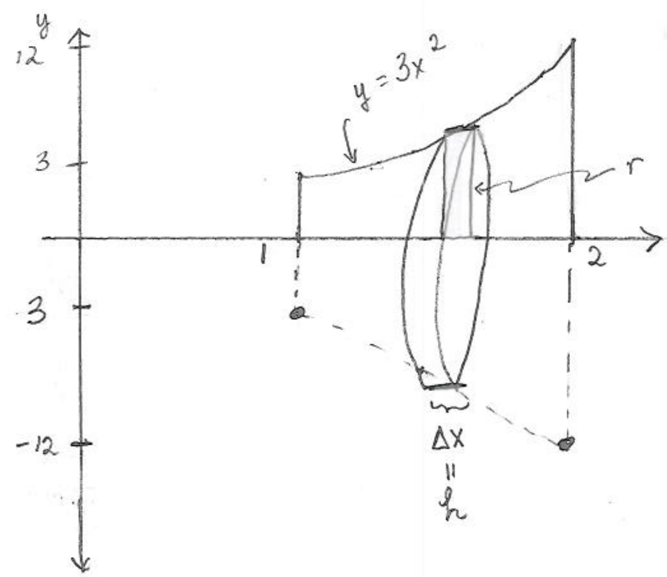
- $R$ will be revolved around some line to create a solid of revolution
- using either the disk, washer, or shell method, express the volume $V$ of the resulting solid of revolution as **one integral** (and NOT as 2 or more integrals).
- In the space provided **below** each problem, make some *good enough sketch* (does not have to be too fancy) to indicate (i.e., help justify) your thinking/reseasoning behind your solution
- you do not have to do lots of algebra to your integrand
- you do not have to integrate your integral.

**Extra Credit/Hint** In the sketch below, draw in a $\boxed{\text{typical rectangle}}$ (should it be horizontal or vertical?) that would be used to express the area of $R$ as precisely 1 integral (and not 2 integrals).




**3a.** The volume $V$ of the solid obtained by revolving the region $R$ about the $x$-axis is

$$V = \int_{x=1}^{x=2} \pi\left(3x^2\right)^2\,dx \quad \underline{\underline{\text{or}}} \quad 9\pi\int_{x=1}^{x=2} x^4\,dx$$




Partition $x$-axis $\Rightarrow$ everything in terms of $x$ $\Rightarrow$ Disk Method

Volume of typical element

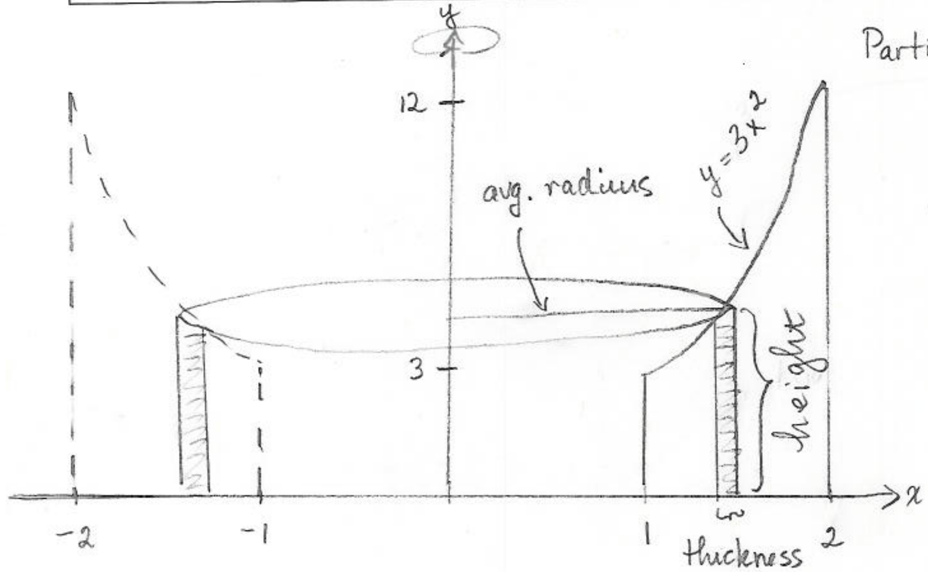
$= (\text{area base})(\text{height})$

$= (\pi r^2)\; h$

$= \pi\left(3x^2\right)^2\; \Delta x$

**3b.** The volume $V$ of the solid obtained by revolving the region $R$ about the $y$-axis is

$$\boxed{V = \int_{x=1}^{x=2} 2\pi\, x\,(3x^2)\, dx \ \overset{\text{or}}{=}\ 6\pi \int_{x=1}^{x=2} x^3\, dx}$$




Partition $x$-axis $\Rightarrow$ everything is in terms of $x$
$\Rightarrow$ Shell Method

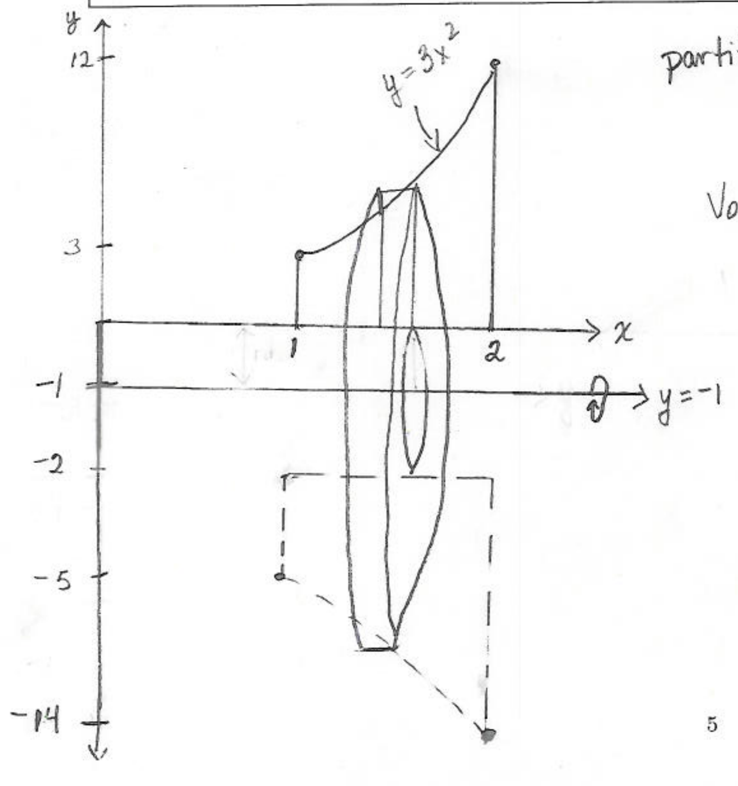
Volume of typical element

$= 2\pi\,(\text{avg. radius})(\text{height})(\text{thickness})$

$= 2\pi\,(x)(3x^2)\,(\Delta x)$

**3c.** The volume $V$ of the solid obtained by revolving the region $R$ about the line $y = -1$ is

$$\boxed{V = \int_{x=1}^{x=2} \pi\left[(3x^2+1)^2 - 1\right] dx \ \overset{\text{or}}{=}\ \pi \int_{x=1}^{x=2} (9x^4 + 6x^2)\, dx}$$




partition $x$-axis $\Rightarrow$ everything in terms of $x$

$\Rightarrow$ Washer method.

Volume of typical element

$= \text{Volume big} - \text{Volume little}$

$= \pi\,(\text{radius}_{\text{big}})^2\,(\text{height}) - \pi\,(\text{radius}_{\text{little}})^2\,(\text{height})$

$= \pi\,(3x^2+1)^2\,(\Delta x) - \pi\,(1)^2\,(\Delta x)$

$= \pi\left[(3x^2+1)^2 - 1\right] dx$

**4.** Express the arclength of the parameterized curve

$$\begin{aligned} x(t) &= t^3 \\ y(t) &= 5t \end{aligned}$$

from the point $P = (1, 5)$ to the point $Q = (8, 10)$ as an integral with respect to $t$

$$\boxed{\text{arclength} = \int_{t=1}^{t=2} \sqrt{(3t^2)^2 + (5)^2}\, dt \ \underline{\underline{\text{or}}}\ \int_{t=1}^{t=2} \sqrt{9t^4 + 25}\, dt}$$

Once again, you do not have to do lots of algebra to your integrand nor integrate your integral.

$$P = (1,5) \iff \begin{bmatrix} 1 = t^3 \\ 5 = 5t \end{bmatrix} \iff t = 1$$

$$Q = (8,10) \iff \begin{bmatrix} 8 = t^3 \\ 10 = 5t \end{bmatrix} \iff t = 2$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^3) = 3t^2$$

$$\frac{dy}{dt} = \frac{d}{dt}(5t) = 5$$

$$\text{arclength} = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

**5.** $$\boxed{\int \frac{\sin(3x)}{17+\cos(3x)}\,dx = -\frac{1}{3}\ln|17+\cos(3x)|+C \text{ or } -\frac{1}{3}\ln(17+\cos(3x))} + C$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

why don't we need the absolute value sign?

$$\boxed{\begin{aligned} u &= 17+\cos(3x) \\ du &= -3\sin(3x)\,dx \end{aligned}}$$

$$\int \frac{\sin(3x)}{17+\cos(3x)}\,dx = \int \frac{1}{17+\cos(3x)}\left[\sin(3x)\,dx\right]$$

$$= \int \frac{1}{u}\left[-\frac{1}{3}\,du\right]$$

$$= -\frac{1}{3}\int \frac{du}{u}$$

$$= -\frac{1}{3}\ln|u| + c$$

$$= -\frac{1}{3}\ln|17+\cos 3x| + c$$

6. $$\boxed{\int \frac{e^x}{\sqrt{1 - e^{(2x)}}}\,dx = \sin^{-1}(e^x) + c}$$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$$

Recall $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

$$\int \frac{e^x}{\sqrt{1-e^{(2x)}}}\,dx = \int \frac{1}{\sqrt{1-(e^x)^2}}\left[e^x\,dx\right]$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}\frac{u}{1} + c$$

$$= \sin^{-1}(e^x) + c$$

**7.** $\boxed{\int x^3\, 17^{(x^4)}\, dx = \dfrac{17^{(x^4)}}{4 \ln 17} + C}$

Remark: box your substitution box for more partial credit.

Recall: you can check your answer via differentiation (if you have time).

Recall $\int a^u\, du = \dfrac{a^u}{\ln a} + c$

$$\boxed{\begin{aligned} u &= x^4 \\ du &= 4x^3\, dx \end{aligned}} \Rightarrow \frac{1}{4}\, du = x^3\, dx$$

$$\int x^3\, 17^{(x^4)}\, dx = \int 17^{(x^4)} \left[x^3\, dx\right]$$

$$= \int 17^u \left[\tfrac{1}{4}\, du\right]$$

$$= \frac{1}{4} \int 17^u\, du$$

$$= \frac{1}{4}\, \frac{17^u}{\ln 17} + c$$

$$= \frac{17^{(x^4)}}{4 \ln 17} + C$$

**8.** Consider a right square pyramid, as sketched in (i) below, of height $h$ where:

- the base of the pyramid is a square with side length $l$
- the line segment connecting
  - the center $C$ of the square base
  - the top point $A$ of the pyramid (which is called the *apex*)

  forms a right angle with the square base and has length $h$ .

Sketches (ii) and (iii) are for hints.

**8a.** In sketch (iii), $C = (\underline{\ 0\ }, \underline{\ 0\ })$ and $A = (\underline{\ 0\ }, \underline{\ h\ })$ and $P = (\underline{\ \frac{l}{2}\ }, \underline{\ 0\ })$

and the equation of the line $L$ is $\underline{y = \frac{-2h}{l}x + h \ \boxed{\text{or}}\ x = \frac{l}{2h}(h - y)}$ .

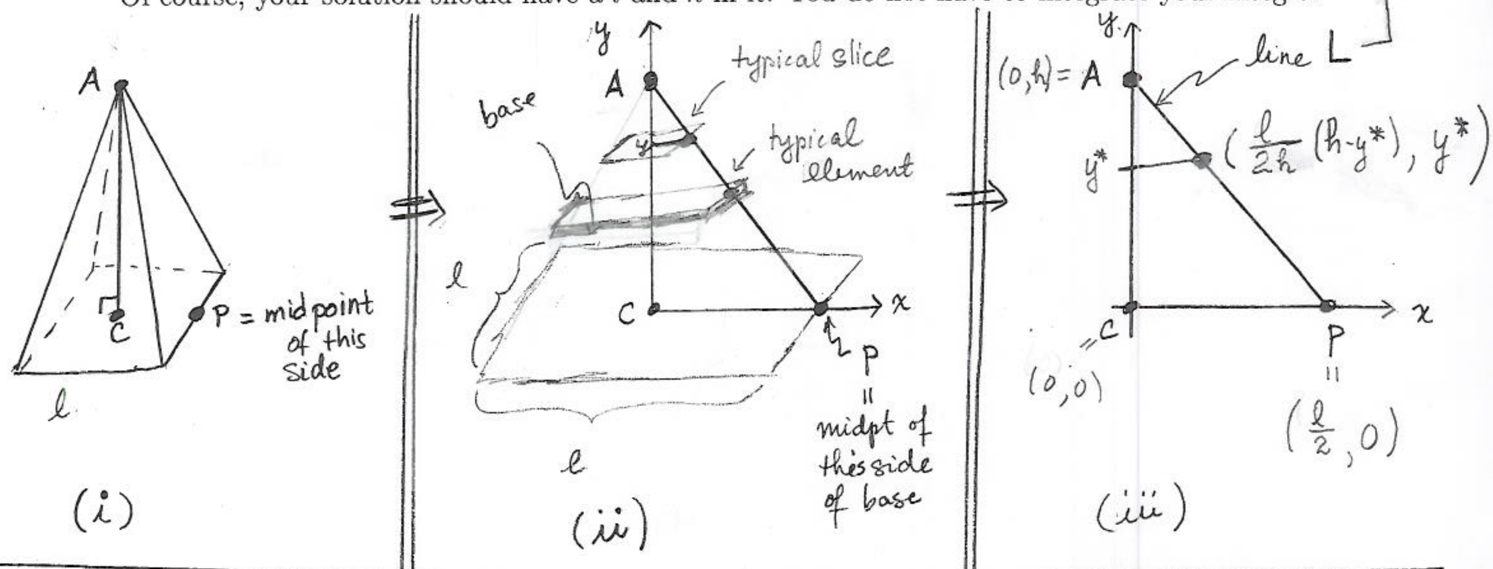
$y = mx + b$

slope $= m = \frac{\Delta y}{\Delta x} = \frac{0 - h}{l/2 - 0} = -\frac{2h}{l}$

**8b.** Using the slicing method, express the volume of the pyramid as an integral.

$$\text{volume} = \int_{y=0}^{y=h} \frac{l^2}{h^2}(h-y)^2\,dy$$

Of course, your solution should have a $l$ and $h$ in it. You do not have to integrate your integral.




Area of typical slice (at $y = y^*$) $= [\text{length of side of square}]^2 = \left[2\left(\frac{l}{2h}(h-y^*)\right)\right]^2$

$= \left[\frac{l}{h}(h-y^*)\right]^2 = \frac{l^2}{h^2}(h-y^*)^2$

Volume of typical element (at $y = y^*$) $= [\text{area of typical slice (at } y = y^*)]\,[\text{height}]$

$= \frac{l^2}{h^2}(h-y^*)^2\,\Delta y$

$$\text{Volume} = \int_{y=0}^{y=h} \frac{l^2}{h^2}(h-y)^2\,dy \overset{\text{fun}}{=} \frac{l^2}{h^2}\int_{y=0}^{y=h}(y-h)^2\,dy = \frac{l^2}{h^2}\left.\frac{(y-h)^3}{3}\right|_{y=0}^{y=h}$$

$$= \frac{l^2}{3h^2}\left[0^3 - (-h)^3\right] = \frac{1}{3}l^2 h = \frac{1}{3}(\text{area base})(\text{height})$$