Prof. Girardi	Math 142	Fall 2006	11.30.06	Exam 3 - Part 2

MARK BOX		
PROBLEM	POINTS	
1 a - j	30	
2	5	
3	10	
4	10	
5 ab	10	
6	10	
7	10	
8	5	
9 - Part 1	10	
%	100	

NAME: Solutions.

please check the box of your section

☐ **Section 005 (WF 8:00 am)**

or

☐ **Section 006 (WF 9:05 am)**

INSTRUCTIONS:

(1) To receive credit you must:
   (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
   (b) if a line/box is provided, then:
      – show you work BELOW the line/box
      – put your answer on/in the line/box
   (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): Part 1: Sections 10.1, 10.2 and Part 2: Sections 10.3, 10.4, 10.5, 10.6 and 10.8 .

Problem Inspiration:

1. you were warned, from class handouts and old exams
2. homework problem § Ch 10 Review # 9 , homework problem § 10.4 # 28
3. Example from class lecture
4. Serious Series Problems # 10
5. homework problem § 10.6 # 29
6. homework problem § 10.8 # 29
7. from textbook § 10.8 # 49
8. homework problem § 10.8 # 63

1. Fill-in-the blanks/boxes. All series $\sum$ are understood to be $\sum_{n=1}^{\infty}$.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1a. n^{th}**-term test** for an arbitrary series $\sum a_n$.

If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum a_n$ diverges.

1b. Geometric Series where $-\infty < r < \infty$. The series $\sum r^n$

- converges if and only if $|r|$ < 1
- diverges if and only if $|r|$ ≥ 1

1c. p**-series** where $0 < p < \infty$. The series $\sum \frac{1}{n^p}$

- converges if and only if p > 1
- diverges if and only if p ≤ 1

1d. Integral Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $f\colon [1,\infty) \to \mathbb{R}$ be so that

or decreasing

- $a_n = f($ n $)$ for each $n \in \mathbb{N}$
- f is a positive function
- f is a continuous function
- f is a nonincreasing function .

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges.

1e. Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1f. Limit Comparison Test for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If $0 < L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges

1g. Ratio and Root Tests for a positive-termed series $\sum a_n$ where $a_n \geq 0$.

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ or $\rho = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$.

- If ρ < 1 then $\sum a_n$ converges.
- If ρ > 1 then $\sum a_n$ diverges.
- If ρ $= 1$ then the test is inconclusive.

1h. Alternating Series Test for an alternating series $\sum(-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

If

- a_n $>$ a_{n+1} for each $n \in \mathbb{N}$
- $\lim_{n\to\infty} a_n =$ 0

then $\sum(-1)^n a_n$ converges

1i. By definition, for an arbitrary series $\sum a_n$, (fill in the blanks with converges or diverges).

- $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ Converges
- $\sum a_n$ is conditionally convergent if and only if $\sum a_n$ converges and $\sum |a_n|$ diverges
- $\sum a_n$ is divergent if and only if $\sum a_n$ diverges

1j. Fill in the 3 blank lines (with absolutely convergent, conditional convergent, or divergent) on the following FLOW CHART for class used to determine if a series $\sum_{n=17}^{\infty} a_n$ is: absolutely convergent, conditional convergent, or divergent.

Does $\sum |a_n|$ converge? Since $|a_n| \geq 0$, use a positive term test: integral test, CT, LCT, ratio/root test.

if NO $\Rightarrow$ Does $\lim_{n\to\infty} |a_n| = 0$?

if NO $\Rightarrow$ $\sum a_n$ is divergent

if YES $\Downarrow$ (from first box)

$\sum a_n$ is absolutely convergent

if YES $\Downarrow$ (from second box)

Is $\sum a_n$ an alternating series?

if YES $\Downarrow$

Does $\sum a_n$ satisfy the conditions of the Alternating Series Test?

if YES $\Downarrow$

$\sum a_n$ is conditionally convergent

2. Circle T if the statement is TRUE. Circle F if the statement if FALSE.

T   F   If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges

T   F   If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

T   F   If $\sum a_n$ converges and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges.

eg, take $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$

T   F   If $\sum(a_n + b_n)$ converges, then $\sum a_n$ converges and $\sum b_n$ converge.

T   F   If $S_N = \sum_{n=1}^{N} r^n$, then $S_N = \dfrac{r - r^{N+1}}{1-r}$.

one, not zero.

$S_N = r + r^2 + \dots + r^N$

$rS_N = \quad r^2 + \dots + r^N + r^{N+1}$

$(1-r)S_N = r - r^{N+1}$

3. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=17}^{\infty} \frac{(-1)^n}{n}$$

- [ ] absolutely convergent
- [x] conditionally convergent
- [ ] divergent

abs conv. ?

$|a_n| = \frac{1}{n}$ p-series

$p=1$

so $\sum |a_n| \to$ divg

$\Rightarrow$ not abs. conv.

cond conv ?

$$\sum_{n=17}^{\infty} (-1)^n \left(\frac{1}{n}\right)$$

a_n AST

✓ is $\lim_{n\to\infty} \frac{1}{n} = 0$ yes

✓ is a_n dec?

derivative $\rightarrow$

$n^{-1} = -1n^{-2} < 0$

yes

so $\rightarrow$ cond. conv.

LCT - WAY

4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$$

- [x] absolutely convergent
- [ ] ~~conditionally convergent~~ $a_n > 0$
- [ ] divergent

$$a_n = \frac{1}{\sqrt{n^3+3n^2+2n}} \overset{n\to big}{\approx} \frac{1}{(n^3)^{\frac{1}{2}}} = b_n$$

$$\lim_{n\to\infty} \frac{(n^3)^{\frac{1}{2}}}{(n^3+3n^2+2n)^{\frac{1}{2}}} = \frac{1}{1} = 1 \qquad 0 < L < \infty$$

$b_n = \left(\frac{1}{n}\right)^{3/2}$ p-series $p >$ converg $p = 3/2$

more algebra details

since Σb_n converges then Σa_n converges by LCT

$$\lim_{n\to\infty} \frac{(n^3)^{\frac{1}{2}}}{(n^3+3n^2+2)^{1/2}} = \lim_{n\to\infty} \left[\frac{n^3}{n^3+3n^2+2}\right]^{1/2}$$

$$= \left[\lim_{n\to\infty} \frac{n^3}{n^3+3n^2+2}\right]^{1/2} = \left[\frac{1}{1}\right]^{1/2} = 1$$

CT-WAY $n(n+1)(n+2)$ $n^2+2n+1n+2$ $n(n^2+3n+2)=\boxed{n^3+3n^2+2n}$

4. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$$

- [x] absolutely convergent
- [ ] ~~conditionally convergent~~ b/c pos. series
- [ ] divergent

$n(n+1)(n+2)$

$n(n^2+3n+2)$

n^3+3n^2+2n

COMPARISON TEST

$b_n = \frac{1}{n^{3/2}}$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n^2+2n}}$$

comp. test

$$a_n = \frac{1}{\sqrt{n^3+3n^2+2n}} \qquad b_n = \frac{1}{n^{3/2}}$$

$\frac{3}{2} > 1$

$\frac{1}{\sqrt{n^3+3n^2+2n}} < \frac{1}{n^{3/2}}$ — by p-series this converges

therefore since $\frac{1}{\sqrt{n^3+3n^2+2n}} < \frac{1}{n^{3/2}}$ then

a_n also converges by comparison test

it converges ABSOLUTELY

5. Let

$$a_n = \frac{n!}{(2n-1)!}$$

5a. Find an expression for $\frac{a_{n+1}}{a_n}$ that does NOT have a fractorial sign (that is a ! sign) in it.

$$\boxed{\frac{a_{n+1}}{a_n} = \frac{n+1}{(2n)(2n+1)}}$$

5b. Check the correct box and then indicate your reasoning below. Specifically specify what test(s) you are using. A correctly checked box without appropriate explanation will receive no points.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{(2n-1)!}$$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

abs conv? $2n+2-1$ $|a_n| = \frac{n!}{(2n-1)!}$

Ratio test →

$$a_{n+1} = \frac{(n+1)!}{(2n+1)!}$$

$$a_n = \frac{n!}{(2n-1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(2n-1)!}{(2n+1)!} =$$

$$\frac{\cancel{n!}(n+1)}{\cancel{n!}} \cdot \frac{\cancel{(2n-1)!}}{\cancel{(2n-1)!}(2n)(2n+1)} =$$

$$\frac{n+1}{(2n)(2n+1)}$$

$$\lim_{n\to\infty} \frac{\overset{0+0}{n+1}}{4n^2+2n} = 0 < 1$$

converges

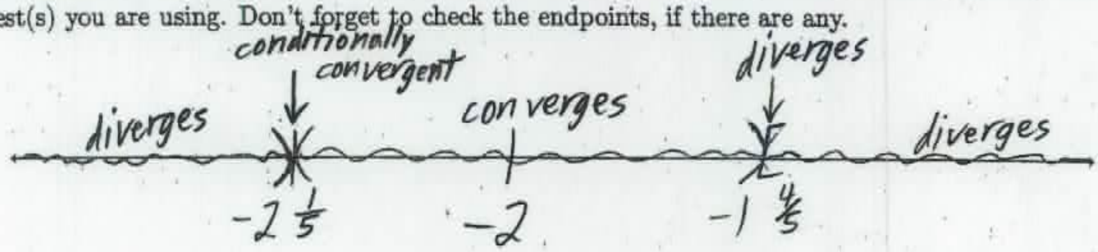
6. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(5x+10)^n}{n}$$

Hint: $(5x+10)^n = [5(x+2)]^n = 5^n (x+2)^n = 5^n (x-(-2))^n$

The center is $x_0 =$ -2 and the radius of convergence is $R =$ $\frac{1}{5}$.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



RT

$$\lim_{n\to\infty} \left| \frac{(5x+10)^{n+1}}{(5x+10)^n} \cdot \frac{n}{(n+1)} \right| = \lim_{n\to\infty} \frac{5^{n+1}|x-(-2)|^{n+1}}{5^n |x-(-2)|^n} \cdot \frac{n}{(n+1)} =$$

$$\lim_{n\to\infty} \frac{\cancel{5^n} \cdot 5 \cdot |x-(-2)| \cdot \cancel{|x-(-2)|^n}}{\cancel{5^n} \cdot \cancel{|x-(-2)|^n}} \cdot \frac{n}{(n+1)} = \lim_{n\to\infty} \frac{5 \cdot |x-(-2)| \cdot n}{(n+1)} =$$

$$5 \cdot |x-(-2)| \lim_{n\to\infty} \frac{n}{n+1} = 5 \cdot |x-(-2)| \cdot 1 = 5 \cdot |x-(-2)|$$

$$5 \cdot |x-(-2)| < 1$$

$$|x-(-2)| < \frac{1}{5}$$

Endpoints

$$\sum_{n=1}^{\infty} \frac{(5(-\frac{9}{5})+10)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \quad p\text{-series, } p=1\text{, diverges}$$

$$\sum_{n=1}^{\infty} \frac{(5(\frac{-11}{5})+10)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ ← problem 3 of the exam

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad p\text{-series, } p=1\text{, diverges}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad ① \ a_n > a_{n+1} \quad ② \ \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\sum a_n$ converges by A

cente = 0

7. Consider the formal power series

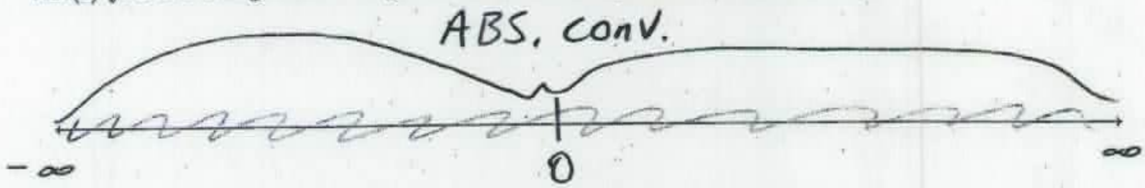
$$\sum_{n=1}^{\infty} \frac{x^n}{(\ln n)^n}.$$

Hint 1: $\frac{x^n}{(\ln n)^n} = \left[\frac{x}{\ln n}\right]^n$ so would you rather use the root test or the ratio test?

Hint 2: $\ln(a^r) = r\ln(a)$ but $(\ln(a))^r \neq r\ln(a)$ +

The center is $x_0 =$ 0 and the radius of convergence is $R =$ ∞.

As we did in class, make a number line indicating where the power series is: absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning and specifically specify what test(s) you are using. Don't forget to check the endpoints, if there are any.



$$\sum_{n=1}^{\infty} \frac{x^n}{(\ln n)^n} \quad \text{ROOT TEST} \quad \lim_{n\to\infty} (a_n)^{1/n} \longrightarrow \lim_{n\to\infty} \left| \left[\frac{x^n}{(\ln n)^n}\right]^{1/n} \right| =$$

$$\lim_{n\to\infty} \left|\frac{x}{\ln n}\right| = |x| \lim_{n\to\infty} \frac{1}{\ln n} = 0 \cdot |x| = 0 < 1 \to \text{converges by Root test}$$

$\therefore$ the interval of convergence is $(-\infty, \infty)$

So radius $= \infty$

$= (x-0)^n$
$= (x-x_0)^n$ $\Rightarrow$ center $= 0$

8. Fill-in the 6 blanks.
Consider the power series

$$\sum_{n=1}^{\infty} (-1)^n a_n x^n$$

where all of the a_n's are positive. Let's say that you know that

if $\quad 0 < x < 17 \quad$ then $\sum (-1)^n a_n x^n$ converges
if $\quad x = 17 \quad$ then $\sum (-1)^n a_n x^n$ conditionally converges
if $\quad 17 < x \quad$ then $\sum (-1)^n a_n x^n$ diverges .

Then this power series has:

center at $x_0 =$ ___0___ and radius of convergence $R =$ ___17___ .

Also, what can you say about the following interval? Fill in the blanks below with:

- is absolutely convergent
- is conditionally convergent
- is divergent
- inconclusive (not enough information given to decide in general).

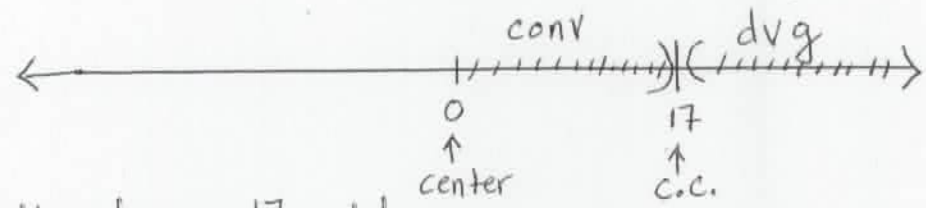
if $\quad -17 < x < 0 \quad$ then $\sum (-1)^n a_n x^n$ ___abs. conv.___

if $\quad x < -17 \quad$ then $\sum (-1)^n a_n x^n$ ___divg.___

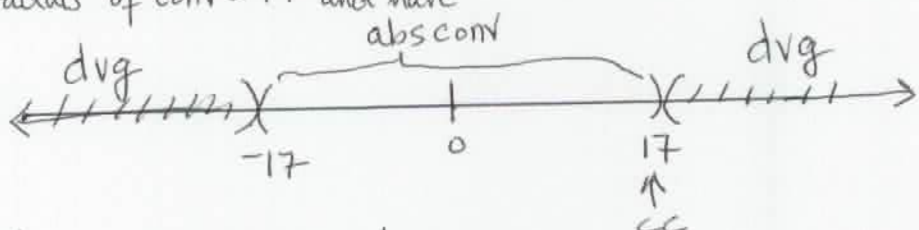
if $\quad x = 0 \quad$ then $\sum (-1)^n a_n x^n$ ___abs. conv.___

if $\quad x = -17 \quad$ then $\sum (-1)^n a_n x^n$ ___divg___

Given



$\Rightarrow$ so radius of conv = 17 and have



What about $x = -17$. Well since $a_n > 0$,

positive b/c a_n's positive

- $\sum (-1)^n a_n x^n \overset{x=17}{=\!=} \sum (-1)^n a_n 17^n = \sum (-1)^n \left[(17)^n a_n\right] \leftarrow$ cc
- $\sum (-1)^n a_n x^n \overset{x=-17}{=\!=} \sum (-1)^n a_n (-17)^n = \sum (-1 \cdot -17)^n a_n = \underbrace{\sum (17)^n a_n}_{\text{dvg b/c}}$