Prof. Girardi	Math 142	Fall 2006	10.25.06	Exam 2

MARK BOX		
PROBLEM	POINTS	
1 a–y	25	
2	10	
3	11	
4	11	
5	11	
6	11	
7	11	
8 take home	10	
9 extra credit take home	2	
%	100	

NAME: Solutions .

please check the box of your section

☐ **Section 005 (WF 8:00 am)**

or

☐ **Section 006 (WF 9:05 am)**

INSTRUCTIONS:

(1) To receive credit you must:
  (a) **work in a logical fashion, show all your work, indicate your reasoning; no credit will be given for an answer that *just appears*;** such explanations help with partial credit
  (b) if a line/box is provided, then:
  — show you work BELOW the line/box
  — put your answer on/in the line/box
  (c) if no such line/box is provided, then box your answer
(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.
(3) You may **not** use a calculator, books, personal notes.
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
(5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8^{th} ed.): Sections 8.2, 8.3, 8.4, 8.5, 8.7, 8.8. .

Problem Inspiration:

1. You were warned.
2. An example from class.
3. Handout of 100 integrals # 7
4. Handout of 100 integrals # 23
5. Handout of 100 integrals # 47
6. Handout of 100 integrals # 26
7. Handout of 100 integrals - 4th page
8. just like the homework.
9. Section 8.8, number 16.

Hints:

(1) **You can check your answers to the indefinite integrals by differentiating.**
(2) **For more partial credit, box your $u - du$ substitutions.**

1. Fill in the blanks (each worth 1 point).

1a. $\int \frac{du}{u} = \underline{\ \ln\ }\ |u| + C$

1b. If a is a constant and $a > 0$ but $a \neq 1$, then $\int a^u\,du = \underline{\ \frac{a^u}{\ln a}\ } + C$

1c. $\int \cos u\,du = \underline{\ \sin u\ } + C$

1d. $\int \sec^2 u\,du = \underline{\ \tan u\ } + C$

1e. $\int \sec u \tan u\,du = \underline{\ \sec u\ } + C$

1f. $\int \sin u\,du = \underline{\ -\cos u\ } + C$

1g. $\int \csc^2 u\,du = \underline{\ -\cot u\ } + C$

1h. $\int \csc u \cot u\,du = \underline{\ -\csc u\ } + C$

1i. $\int \tan u\,du = \underline{\ -\ln|\cos u| + C \text{ or } \ln|\sec u| + C\ }$ $\cancel{+C}$

1j. $\int \cot u\,du = \underline{\ \ln|\sin u| + C \text{ or } -\ln|\csc u| + C\ }$ $\cancel{+C}$

1k. $\int \sec u\,du = \underline{\ \ln|\sec u + \tan u| + C \text{ or } -\ln|\sec u - \tan u| + C\ }$ $\cancel{+C}$

1l. $\int \csc u\,du = \underline{\ -\ln|\csc u + \cot u| + C \text{ or } \ln|\csc u - \cot u| + C\ }$ $\cancel{+C}$

1m. If a is a contant and $a > 0$ then $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \underline{\ \arcsin \frac{u}{a}\ } + C$

1n. If a is a contant and $a > 0$ then $\int \frac{1}{a^2+u^2}\,du = \underline{\ \frac{1}{a} \arctan \frac{u}{a}\ } + C$

1o. If a is a contant and $a > 0$ then $\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \underline{\ \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a}\ } + C$

1p. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polyonomials and [degree of f] $\geq$ [degree of g], then one must first do __long division__

1q. Integration by parts formula: $\int u\,dv = \underline{\ uv - \int v\,du\ }$

1r. Trig substitution: (recall that the *integrand* is the function you are integrating)
if the integrand involves $a^2 - u^2$, then one makes the substitution $u = \underline{\ a \sin\theta\ }$

1s. Trig substitution:
if the integrand involves $a^2 + u^2$, then one makes the substitution $u = \underline{\ a \tan\theta\ }$

1t. Trig substitution:
if the integrand involves $u^2 - a^2$, then one makes the substitution $u = \underline{\ a \sec\theta\ }$

1u. trig formula ... your answer should involve trig functions of θ, and not of 2θ: $\sin(2\theta) = \underline{\ 2 \sin\theta \cos\theta\ }$.

1v. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$.

1w. trig formula ... $\cos(2\theta)$ should appear in the numerator: $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$.

1x. trig formula ... since $\cos^2\theta + \sin^2\theta = 1$, we know that the corresponding relationship beween tangent (i.e., tan) and secant (i.e., sec) is $\underline{\ 1 + \tan^2\theta = \sec^2\theta\ }$.

1y. $\arcsin(-\frac{\sqrt{2}}{2}) = \underline{\ -\pi/4\ }$ **RADIANS**. (your answer should be an angle)

2. $\boxed{\displaystyle\int_{x=17}^{x=\infty} \frac{1}{x^2}\,dx \;=\; \frac{1}{17}}$

$$\int_{x=17}^{x=\infty} x^{-2}\,dx = \lim_{b\to\infty} \int_{x=17}^{x=b} x^{-2}\,dx$$

$$= \lim_{b\to\infty} \frac{x^{-1}}{-1} \Big|_{x=17}^{x=b}$$

$$= \lim_{b\to\infty} \frac{-1}{x} \Big|_{x=17}^{x=b}$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b} - -\frac{1}{17} \right]$$

$$= \lim_{b\to\infty} -\frac{1}{b} + \lim_{b\to\infty} \frac{1}{17}$$

$$= 0 + \frac{1}{17}$$

3.

3a. $D_x \tan x =$ $\sec^2 x$

3b. $D_x \sec x =$ $\sec x \tan x$

3c. $\int \tan^2 x \, dx =$ $\tan x - x$ $+ C$

Hint: problems 1x, 3a, and 3b might come in handy for problem 3c.

$$\int \tan^2 x \, dx \overset{(1x)}{=} \int (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \, dx - \int dx$$

$$\downarrow (3a)$$

$$= \tan x - x + C$$

4. $$\boxed{\int \ln(3x+6)\,dx = (x+2)\ln(3x+6) - x \qquad + C}$$

Hint: be clever in your choice of v to avoid unneeded work.

Note that $\frac{3}{3x+6} = \frac{3}{3(x+2)} = \frac{1}{x+2}$.

Parts - Lesson #4 $f(x) = \ln(3x+6)$ is easy to differentiate but hard to integrate so try letting $u = f(x)$.

Way #1

$$\boxed{\begin{array}{ll} u = \ln(3x+6) & dv = dx \\ du = \frac{3}{3x+6}\,dx & v = x \end{array}}$$

$$3x+6 \overline{\big) 3x + 0}\ \ 1, \quad \frac{3x+6}{-6}$$

$$\int \ln(3x+6)\,dx = x\ln(3x+6) - \int \frac{3x}{3x+6}\,dx$$

$$= x\ln(3x+6) - \int \left(1 - \frac{6}{3x+6}\right)dx$$

$$= x\ln(3x+6) - \int dx + 6\,\frac{1}{3}\int \frac{3\,dx}{3x+6}$$

$$= x\ln(3x+6) - x + 2\ln(3x+6) + C$$

Way #2

$$\boxed{\begin{array}{ll} u = \ln(3x+6) & dv = dx \\ du = \frac{3}{3x+6}\,dx = \frac{1}{x+2}\,dx & v = x+2 \end{array}}$$

$$\int \ln(3x+6)\,dx = (x+2)\ln(3x+6) - \int x+2 \cdot \left(\frac{1}{x+2}\right) dx$$

$$= (x+2)\ln(3x+6) - \int dx$$

$$= (x+2)\ln(3x+6) - x + C$$

5. $$\boxed{\int \frac{x^4+2x+2}{x^5+x^4}\,dx = \frac{-2}{3x^3} + \ln|x+1| \qquad + C}$$

Hint: $x^5+x^4 = (x^4)(x+1) = (x-0)^4\,(x+1)^1$.

$$\frac{x^4+2x+2}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1} \leftarrow \text{Note } x^4 = (x-0)^4$$

$$\Rightarrow \frac{x^4+2x+2}{x^4(x+1)} = \frac{Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4}{x^4(x+1)}$$

$$\Rightarrow x^4+2x+2 = Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4$$

$\boxed{x=0 \Rightarrow 2 = D}$ $\quad$ $\boxed{x=-1 \Rightarrow 1 = E}$

x^4 : $1 = A + E \longrightarrow \boxed{A=0}$

x^3 : $0 = A + B$

x^2 : $0 = B + C \longrightarrow \boxed{B=0}$

x : $2 = C + D \longrightarrow \boxed{C=0}$

constant : $2 = D$

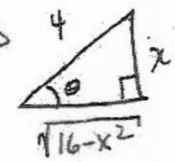
$$\int \frac{x^4+2x+2}{x^5+x^4}\,dx = \int \left(\frac{2}{x^4} + \frac{1}{x+1}\right) dx$$

$$= \int 2x^{-4}\,dx + \int \frac{dx}{x+1} = \boxed{\frac{2x^{-3}}{-3} + \ln|x+1| + c}$$

6. $$\boxed{\int \frac{x^2}{\sqrt{16-x^2}}\,dx = 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} \qquad + C}$$

Hint: problems 1 u - w might come in handy.

$x = 4\sin\theta \quad \Rightarrow \sin\theta = \dfrac{x}{4} \quad \Rightarrow$



$\Rightarrow \cos\theta = \dfrac{\sqrt{16-x^2}}{4}$

$dx = 4\cos\theta\, d\theta$

$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16(1-\sin^2\theta)} = 4\sqrt{\cos^2\theta} = 4\cos\theta$

$$\int \frac{x^2\, dx}{\sqrt{16-x^2}} = \int \frac{(16\sin^2\theta)(4\cos\theta\, d\theta)}{4\cos\theta} = 16\int \sin^2\theta\, d\theta$$

$$\overset{(1w)}{=} \frac{16}{2}\int (1-\cos 2\theta)\, d\theta = 8\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$$

$$\overset{(1u)}{=} 8\theta - 4\sin(2\theta) + C = 8\theta - 4\cdot 2\sin\theta\cos\theta$$

$$= 8\sin^{-1}\left(\frac{x}{4}\right) - 8\left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right) + C$$

7.

$$\boxed{\int \csc^3 x\,dx = \frac{1}{2}\left(-\csc x \cot x + \ln|\csc x - \cot x|\right) \quad + C}$$

Hint: bring to the other side idea, similar to how to do $\int \sec^3 x dx$. Helpful:

- $D_x \cot x = -\csc^2 x$
- $D_x \csc x = -\csc x \cot x$
- $\cot^2 x + 1 = \csc^2 x$

$u = \csc x \qquad dv = \csc^2 x \, dx$

$du = -\csc x \cot x \, dx \qquad v = -\cot x$

$$\int \csc^3 x \, dx = -\csc x \cot x - \int \csc x \cot^2 x \, dx$$

$$= -\csc x \cot x - \int \csc x (\csc^2 x - 1) \, dx$$

$$= -\csc x \cot x - \int \csc^3 x \, dx + \int \csc x \, dx$$

$\Rightarrow$

$$2\int \csc^3 x \, dx = -\csc x \cot x + \int \csc x \, dx$$

$$= -\csc x \cot x - \ln|\csc x + \cot x| + c$$

$$\overset{or}{=} -\csc x \cot x + \ln|\csc x - \cot x| + c$$

8. Numerical Integration . Let

$$I = \int_{x=1}^{x=3} \frac{1}{x^2}\, dx .$$

The 10 steps of this problem are similar to the homework but the number of subintervals is **6** and **not 10. On the parts that say "Do not use a calculator", you need not do alot of arthritic.**

8-1. Find the exact value of I, without using a calculator. Your answer should be a fraction, without decimal places.

$I = \frac{2}{3}$

$$\int_{x=1}^{x=3} x^{-2}\, dx = \frac{x^{-1}}{-1}\Big|_{x=1}^{x=3} = \frac{1}{x}\Big|_{x=3}^{x=1} = 1 - \frac{1}{3} = \frac{2}{3}$$

8-2. Find an approximation for I, using part 1 and your calculator, to as many decimal places as your calculator will give you.

$I \approx .666666666666\bar{6}$

8-3. Approximate I using the Trapezoid Rule T_n with $n = 6$ subintervals. Do not use a calculator (so your answer will have several numbers added together).

$T_6 = \frac{3-1}{2(6)} \left[1\,(1) + 2\left(\frac{3}{4}\right)^2 + 2\left(\frac{3}{5}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{3}{7}\right)^2 + 2\left(\frac{3}{8}\right)^2 + 1\left(\frac{1}{3}\right)^2 \right]$

or $\frac{1}{6}$

Divide $[1,3]$ into 6 subintervals so $\Delta x = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
$a = 1 = \frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3} = 2$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{9}{3} = 3 = b$

8-4. Find an approximation for T_6, using part 3 and your calculator, to as many decimal places as your calculator will give you.

$T_6 \approx .6841180083$

8-5. Approximate I using Simpon's Rule S_{2n} with $2n = 6$ subintervals. Do not use a calculator (so your answer will have several numbers added together).

$$S_6 = \underbrace{\frac{1}{3}\left(\frac{3-1}{6}\right)}\left[1 + 4\left(\tfrac{3}{4}\right)^2 + 2\left(\tfrac{3}{5}\right)^2 + 4\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{3}{7}\right)^2 + 4\left(\tfrac{3}{8}\right)^2 + 1\left(\tfrac{1}{3}\right)^2\right]$$

or $\frac{1}{9}$ or $\frac{1}{3} \cdot \frac{1}{3}$ or ...

8-6. Find an approximation for S_6, using part 5 and your calculator, to as many decimal places as your calculator will give you.

$S_6 \approx .6678842278$

8-*. Find the first 4 derivatives of $f(x) = x^{-2}$.

$f(x) = x^{-2} = x^{-2}$

$f'(x) = -2x^{-3} = -2!\, x^{-3}$

$f^{(2)}(x) = 6x^{-4} = 3!\, x^{-4}$

$f^{(3)}(x) = {}^{-}24x^{-5} = -4!\, x^{-5}$

$f^{(4)} = 120\, x^{-6} = 5!\, x^{-6}$

8-7. Use inequality (11), page 563, to find an upper bound on the error in part 3. Do not use a calculator.

$$|T_6 - I| \le \frac{(3-1)^3}{12 \cdot 6^2}(6) \text{ or } \frac{1}{9}$$

$$K_2 = \max_{a \le x \le b} |f''(x)| = \max_{1 \le x \le 3} |6x^{-4}| = \max_{1 \le x \le 3} \frac{6}{x^4}$$

$$= \frac{6}{1^4} = 6$$

8-8. Use your calculator to approximate the error estimate in part 7 to as many decimal places as your calculator will give you.

$$|T_6 - I| \lessapprox 0.\overline{1}$$

8-9. Use inequality (12), page 563 to find an upper bound on the error in part 5. Do not use a calculator.

$$|S_6 - I| \le \frac{(3-1)^5\,(120)}{180\,(6)^4} \text{ or } \frac{2^2}{3^5} \text{ or } \frac{4}{243}$$

$$K_4 = \max_{a \le x \le b} |f^{(4)}(x)| = \max_{1 \le x \le 3} \left|\frac{120}{x^6}\right| = \max_{1 \le x \le 3} \frac{120}{x} = 120$$

8-10. Use your calculator to approximate the error iestimate in part 9, to as many decimal places as your calculator will give you.

$$|S_6 - I| \lessapprox 0.016460905 3$$

9. Extra Credit - number 16 from section 8.8.

$$\int_{x=-\infty}^{x=\infty} \frac{e^{-x}}{1+e^{-2x}}\,dx = \frac{\pi}{2}.$$

First $u = e^{-x}$

$du = -e^{-x}\,dx$

$$\int \frac{e^{-x}}{1+e^{-2x}}\,dx = -\int \frac{-e^{-x}\,dx}{1+(e^{-x})^2} = -\int \frac{du}{1+u^2} = -\arctan u + c$$

$$= -\arctan e^{-x} + c$$

$$\int_{x=-\infty}^{x=0} \frac{e^{-x}}{1+e^{-2x}}\,dx = \lim_{a\to-\infty} \int_{x=a}^{x=0} \frac{e^{-x}\,dx}{1+e^{-2x}} = \lim_{a\to-\infty} -\arctan e^{-x} \Big|_{x=a}^{x=0}$$

$$= \lim_{a\to\infty} \left(-\arctan e^{0} - -\arctan e^{-a}\right)$$

$$= \lim_{a\to-\infty} \left(-\arctan 1 + \arctan e^{-a}\right)$$

$$= \lim_{a\to-\infty} \left(-\frac{\pi}{4} + \arctan e^{-a}\right)$$

this much is not math. correct - idea behind it all.

$$= -\frac{\pi}{4} + \underbrace{\lim_{a\to-\infty} \arctan e^{-a}}$$

↳ think of as $\arctan e^{-(-\infty)} = \arctan e^{\infty} = \arctan \infty = \frac{\pi}{2}$

$$= -\frac{\pi}{4} + \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

$$\int_{x=0}^{x=\infty} \frac{e^{-x}}{1+e^{-2x}}\,dx = \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{e^{-x}\,dx}{1+e^{-2x}} = \lim_{b\to\infty} -\arctan e^{-x} \Big|_{x=0}^{x=b}$$

$$= \lim_{b\to\infty} \left(-\arctan e^{-b} - -\arctan e^{0} \right)$$

$$= \lim_{b\to\infty} \left(-\arctan e^{-b} + \arctan 1 \right)$$

$$= \lim_{b\to\infty} \left(-\arctan e^{-b} + \frac{\pi}{4} \right)$$

$$= \frac{\pi}{4} + \lim_{b\to\infty} -\arctan e^{-b}$$

not math correct but do you get the idea?

think of as $-\arctan e^{-\infty} = -\arctan 0$
$= -0 = 0$

$$= \frac{\pi}{4} + 0.$$

$$\int_{x=-\infty}^{x=\infty} \frac{e^{-x}}{1+e^{-2x}}\,dx = \int_{x=-\infty}^{x=0} \frac{e^{-x}}{1+e^{-2x}}\,dx + \int_{x=0}^{x=+\infty} \frac{e^{-x}}{1+e^{-2x}}\,dx$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$