1 Find and simplify if necessary

(a) $D_x[e^{3x^2+1}] = 6xe^{3x^2+1}$ since $(e^u)' = u'e^u$.

(b) $D_x[\ln(3x^2+17)] = \frac{6x}{3x^2+17}$ since $(\ln(u))' = \frac{u'}{u}$.

(c) $D_x[(1+x)^{2x}]$.

let $y = (1+x)^{2x}$, then $\ln y = \ln(1+x)^{2x}$ or

$\ln y = 2x\ln(1+x)$. Differentiate both sides with respect to x gives

$$\frac{y'}{y} = 2x \cdot \frac{1}{1+x} + 2\ln(1+x)$$

$$\Leftrightarrow y' = y\left[\frac{2x}{1+x} + 2\ln(1+x)\right] = (1+x)^{2x}\left[\frac{2x}{1+x} + 2\ln(1+x)\right]$$

$$\Rightarrow D_x[(1+x)^{2x}] = (1+x)^{2x}\left[\frac{2x}{1+x} + 2\ln(1+x)\right].$$

(d) $D_x[\sin^3(4x)] = 3\sin^2(4x)\cdot(\sin(4x))'$

$= 3\sin^2(4x)\cos(4x)\cdot 4$

$= 12\sin^2(4x)\cos(4x)$.

(e) $\frac{d}{dx}e^{\tan x} = (\tan x)'e^{\tan x} = \sec^2 x e^{\tan x}$.

(f) $\frac{d}{dx}[\ln x]^{2x+3}$

let $y = (\ln x)^{2x+3} \Leftrightarrow \ln y = \ln(\ln x)^{2x+3}$

$\Leftrightarrow \ln y = (2x+3)\ln(\ln x)$. Differentiate both sides

$$\frac{y'}{y} = (2x+3)\left[\frac{(\ln x)'}{\ln x}\right] + 2\ln(\ln x) = (2x+3)\left(\frac{1}{x\ln x}\right) + 2\ln(\ln x)$$

$$\Rightarrow y' = y\left[\frac{2x+3}{x\ln x} + 2\ln(\ln x)\right]$$

So, $$\frac{d}{dx}[\ln x]^{2x+3} = (\ln x)^{2x+3}\left[\frac{2x+3}{x\ln x} + 2\ln(\ln x)\right]$$

(g) $$D_x\left[17^{3x^2+1}\right] = 6x \cdot 17^{3x^2+1}\ln(17)$$

(h) $$D_x[\ln(\cos(4x))]$$
$$= \frac{(\cos 4x)'}{\cos 4x} = \frac{-4\sin 4x}{\cos 4x}.$$

---

2 Integrate each of the following using the appropriate method.

(a) $\int \ln x\, dx$. This was done in class.

(b) $\int \sin^2 x\, dx$. Done in class.

(c) $\int \sin^3 x\, dx$. Same as above.

(d) $\int x^2 \sin x\, dx$

let $u = x^2 \Rightarrow du = 2x\,dx$ and $dv = \sin x\, dx$

$\Updownarrow$

$v = -\cos x$

$$= -x^2\cos x + 2\int x\cos x\, dx \quad \text{------} \; (1)$$

we need to do by part again for $\int x\cos x\, dx$

Let $u = x \Leftrightarrow du = dx$ and $dv = \cos x\,dx$
$\Leftrightarrow v = \sin x$

So, $\int x\cos x\,dx = x\sin x - \int \sin x\,dx$
$= x\sin x + \cos x + C_1$

Hence, $\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$

$$= -x^2\cos x + 2x\sin x + 2\cos x + 2C_1$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + C.$$

(e) $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$

Since the degree of numerater is greater than that of the denominator, so we perform a long division.

$$\begin{array}{r|l} & x + 1 \\ \hline x^3 - x^2 - x + 1 & x^4 - 2x^2 + 4x + 1 \\ & x^4 - x^3 - x^2 + x \\ \hline & x^3 - x^2 + 3x + 1 \\ & x^3 - x^2 - x + 1 \\ \hline & 4x \end{array}$$

$$= \int \left(x + 1 + \frac{4x}{x^3 - x^2 - x + 1} \right) dx$$

Notice that $x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x-1)(x^2-1)$
$= (x-1)(x-1)(x+1) = (x+1)(x-1)^2$

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$4x = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

- when $x=1$, $4 = 2C \Rightarrow C = 2$
- when $x=-1$, $-4 = 4A - 2C \Rightarrow A = 0$
- when $x=0$, $0 = A - B + C \Rightarrow B = 2$

So, $\int \left(x+1 + \frac{2}{x-1} + \frac{2}{(x-1)^2}\right) dx$

$$= \int x\,dx + \int 1\,dx + 2\int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2}$$

$$= \frac{x^2}{2} + x + 2\ln|x-1| - \frac{2}{x-1} + K.$$

(f) $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$. This can be done by u-substitution. let $u = 1-x^2 \Leftrightarrow du = -2x\,dx$

$$\Updownarrow$$

$$x^2 = 1-u$$

rewrite it as

$$= \int \frac{x^2 \cdot x\,dx}{\sqrt{1-x^2}} = \frac{-1}{2}\int \frac{x^2 \cdot -2x\,dx}{\sqrt{1-x^2}}$$

$$= \frac{-1}{2}\int \frac{(1-u)\,du}{\sqrt{u}} = -\frac{1}{2}\int (1-u)u^{-1/2}\,du$$

$$= -\frac{1}{2}\int \left(u^{-1/2} - u^{1/2}\right) du$$

$$= -\frac{1}{2}\int u^{-1/2}\,du + \frac{1}{2}\int u^{1/2}\,du$$

$$= -\frac{1}{2}\cdot\frac{u^{1/2}}{1/2} + \frac{1}{2}\,\frac{u^{3/2}}{3/2} + t$$

$$= -\sqrt{u} + \frac{1}{3}u\sqrt{u} + t$$

$$= -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)\sqrt{1-x^2} + t.$$

(9) $\int x^2 \arctan x\,dx$. We can use the trick

LIATE.

This tells us to choose $u = \arctan x \Leftrightarrow du = \frac{dx}{1+x^2}$

and $dv = x^2 dx \Leftrightarrow v = \frac{x^3}{3}$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int \frac{x^3\,dx}{1+x^2}$$

long division

$$\begin{array}{r} x \\ 1+x^2 \,\big)\, \overline{x^3} \\ x^3+x \\ \hline -x \end{array}$$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2}\right)dx$$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int \frac{x\,dx}{1+x^2}$$

$u = 1+x^2$
$du = 2x\,dx$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\,\frac{x^2}{2} + \frac{1}{6}\ln(1+x^2) + C.$$

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6} + \frac{\ln(1+x^2)}{6} + C.$$

(h) $\int e^x \cos x \, dx$

This is Example 5 on page 516.

(i) $\displaystyle\int \frac{x}{x^4+4x^2+8}\,dx$

rewrote 8 as 4 + 4

$x^4+4x^2+8 = \underbrace{x^4+4x^2+4}+4$

by completing the square method $= (x^2+2)^2+4 = 4+(x^2+2)^2$

$= 4\left[1+\left(\frac{x^2+2}{2}\right)^2\right]$.

let $\frac{x^2+2}{2} = u \Leftrightarrow \frac{2x}{2}dx = du \Leftrightarrow x\,dx = du$

$\displaystyle = \int \frac{du}{4[1+u^2]} = \frac{1}{4}\int \frac{du}{1+u^2} = \frac{1}{4}\operatorname{Arctan}(u)+C$

$= \frac{1}{4}\operatorname{Arctan}\left(\frac{x^2+2}{2}\right) + C.$

(j) $\displaystyle\int \frac{x^4+2x+2}{x^5+x^4}\,dx$ $\qquad x^5+x^4 = x^4(x+1)$

$\dfrac{x^4+2x+2}{x^4(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x^4} + \dfrac{E}{x^5}$

This is only the set-up. You can finish it by partial fractions.

(K) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

We will do it by trig-substitution.

$x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

then $4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta$

$dx = 2\cos\theta \, d\theta$

$$= \int \frac{4\sin^2\theta \cdot \cancel{2\cos\theta}\, d\theta}{\cancel{2\cos\theta}}$$

$$= 4\int \sin^2\theta \, d\theta = 4\int \frac{1-\cos 2\theta}{2}\, d\theta$$

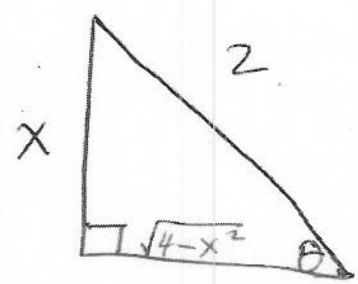
$$= 2\int d\theta - 2\int \cos 2\theta \, d\theta = 2\theta - \frac{2}{2}\sin(2\theta) + C$$

$$= 2\theta - \sin(2\theta) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \cancel{2}\left(\frac{x}{\cancel{2}}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2} + C.$$



$x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2}$

$\Updownarrow$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

(L) $\int_0^\infty \frac{dx}{1+x} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x}$

$= \lim_{t\to\infty} \ln|1+x| \Big|_0^t = \lim_{t\to\infty} \left(\ln|1+t| - \cancel{\ln|1|}^{\,0}\right)$

$= \ln(\infty) = \infty.$

(m) $\int_{-\infty}^{\infty} \frac{x}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{x}{x^2+1}\,dx + \int_0^\infty \frac{x\,dx}{x^2+1}$

$= \lim_{a\to-\infty} \int_a^0 \frac{x\,dx}{x^2+1} + \lim_{b\to\infty} \int_0^b \frac{x\,dx}{x^2+1}$

$= \lim_{a\to-\infty} \frac{1}{2}\ln(x^2+1)\Big|_a^0 + \lim_{b\to\infty} \frac{1}{2}\ln|x^2+1|\Big|_0^b$

let $u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\int \frac{du}{u}$
$= \frac{1}{2}\ln|u| + C$

$= \lim_{a\to-\infty}\left(\cancel{\tfrac{1}{2}\ln(1)}^{\,0} - \tfrac{1}{2}\ln(a^2+1)\right) +$

$\lim_{b\to\infty}\left(\tfrac{1}{2}\ln(b^2+1) - \cancel{\tfrac{1}{2}\ln(1)}^{\,0}\right)$

$= -\frac{1}{2}\ln(\infty) + \frac{1}{2}\ln(\infty) = \infty.$

$\boxed{3}$ Find the limit.

(a) $\lim_{x\to\infty} x^{1/x} \longrightarrow \infty^{\frac{1}{\infty}} = \infty^0$ indeterminate

let $y = x^{1/x} \Longleftrightarrow \ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$

So, $\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

$$\overset{\text{l'Hopital's Rule}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0.$$

So, $\ln(y) = 0 \Rightarrow y = e^0 = 1.$

Hence $\lim_{x\to\infty} x^{1/x} = 1.$

(b) $$\lim_{n\to\infty} \frac{12n^{17}+188n^7-19n}{4n^{18}-n^9+10} = \lim_{n\to\infty} \frac{\frac{12}{n}+\frac{188}{n^{9}}-\frac{19}{n^{17}}}{4-\frac{1}{n^9}+\frac{10}{n^{18}}}$$

$$= \frac{0}{4} = 0.$$

(c) $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$, $c \neq 0$

We can use the fact that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$

But How??, we let $\frac{c}{x} = \frac{1}{y}$ or $y = \frac{x}{c}$.

Then as $x\to\infty$, $y\to\infty$ as well.

So, $\lim_{x \to \infty} (1+\frac{c}{x})^x = \lim_{y \to \infty} (1+\frac{1}{y})^{cy}$

$= \lim_{y \to \infty} \left[(1+\frac{1}{y})^y\right]^c = \left[\lim_{y \to \infty} (1+\frac{1}{y})^y\right]^c$

$= e^c$.

(d) $\lim_{n \to \infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+ - - - - +\sqrt{n}}{n^{3/2}}$ use the fact that $1+2+3+ - - - - - + n = \frac{n(n+1)}{2}$.

So, $\sqrt{1}+\sqrt{2}+\sqrt{3}+ - - - - - + \sqrt{n} = \frac{\sqrt{n}(\sqrt{n}+1)}{2}$

and hence the given limit equals to

$$\frac{\frac{\sqrt{n}(\sqrt{n}+1)}{2}}{n^{3/2}} = \lim_{n \to \infty} \frac{n^{1/2}(n^{1/2}+1)}{2n^{3/2}}$$

$$= \lim_{n \to \infty} \frac{n+n^{1/2}}{2n^{3/2}} = \lim_{n \to \infty} \frac{\frac{1}{n^{1/2}}+\frac{1}{n}}{2} = 0.$$

4 Decide if the given series diverges, converges conditionally, or converges absolutely.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

let $a_n = \frac{1}{\sqrt{n}} \Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ and

$a_{n+1} = \frac{1}{\sqrt{n+1}}$, so $a_n > a_{n+1}$ $\forall n \geq 1$.

So, the two conditions for AST are satisfied.

Hence it converges.

$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ which diverges by p-Test ($p = 1/2 < 1$).

Therefore, the given series converges conditionally.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{(3^n)n!}{(2n)!}$

let $u_n = \frac{(3^n)n!}{(2n)!} \Rightarrow u_{n+1} = \frac{(3^n \cdot 3^1)(n+1)!}{(2n+2)!}$

So, $\rho = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{3^n \cdot 3^1 (n+1) n!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{3^n \cdot (n)!} \right|$

$= \lim_{n\to\infty} \left| \frac{3(n+1)}{2(n+1)(2n+1)} \right| = \lim_{n\to\infty} \left| \frac{3}{2(2n+1)} \right| = 0 < 1$

So, by the ratio test for absolute convergence, the given series converges absolutely.

(c) $\sum_{n=7}^{\infty} (-1)^n \frac{n}{1+n^2}$, let $a_n = \frac{n}{1+n^2}$

$\lim_{n\to\infty} a_n = 0$ and let $f(x) = \frac{x}{1+x^2}$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} \leq 0 \quad \forall\ x \geq 7.$$

Hence by AST, it converges.

$$\sum_{n=7}^{\infty} \left| \frac{(-1)^n n}{1+n^2} \right| = \sum_{n=7}^{\infty} \frac{n}{1+n^2}$$

Since we showed $f'(x) < 0 \ \forall\ x \geq 7$, the integral test is applicable.

$$\int_7^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_7^b \frac{x}{1+x^2}\,dx = \frac{1}{2} \lim_{b\to\infty} \int_7^b \frac{2x\,dx}{1+x^2}$$

$$= \frac{1}{2} \lim_{b\to\infty} \ln(1+x^2) \Big|_7^b$$

$$= \frac{1}{2} \lim_{b\to\infty} \left[\ln(1+b^2) - \ln(50) \right]$$

$$= \frac{1}{2} \left[\infty - \ln(50) \right] = \infty.$$ So, it diverges.

Hence, the given series converges conditionally.

$\boxed{5}$ Find the 3^{rd} order Talyor polynomial of $f(x) = (1+x)^{3/2}$ about $a=0$.

Solution:

$f'(x) = \frac{3}{2}(1+x)^{1/2}$ and $f''(x) = \frac{3}{4}(1+x)^{-1/2}$ and

$f'''(x) = -\frac{3}{8}(1+x)^{-1}$

So, $P_3(x) = f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \frac{f'''(0)x^3}{3!}$

$= 1 + \frac{3}{2}x + \frac{3}{4}\cdot\frac{x^2}{2!} + \frac{3}{8}\cdot\frac{x^3}{3!}$

$\Rightarrow P_3(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3.$

$\boxed{6}$ Let $f(x) = \ln(2+x)$, let $a=0$ and $f(x) = P_3(x) + R_3(x)$, where P_3 is the 3^{rd} order Taylor polynomial of f about a and R_3 is the corresponding remainder term.

(a) Find a form for $R_3(x)$. Your answer should have a "c" in it, be sure to indicate where c lies.

(b) Find a good upper bound for $|R_3(0.5)|$;
i.e. $|R_3(0.5)| \le$?

(a) In order to find $R_3(x)$, we first need to fi.
$P_3(x)$.

$$f(x) = \ln(2+x) \Rightarrow f'(x) = \frac{1}{2+x} \Rightarrow f''(x) = \frac{-1}{(2+x)}$$

and $$f'''(x) = \frac{2}{(2+x)^3}$$

So, $$P_3(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= \ln(2) + \frac{1}{2}x - \frac{1}{2 \cdot 2!}x^2 + \frac{1}{4} \cdot \frac{1}{3!}x^3$$

$$\Rightarrow P_3(x) = \ln(2) + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{24}.$$

So, $$R_3(x) = f(x) - P_3(x)$$

$$\Rightarrow R_3(x) = \ln(2+x) - \ln(2) - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{24}.$$

Don't worry about the "c".

We know that $|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$

$x_0 = a = 0, \quad n = 3.$

I will cover this part in class.

7 Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n}.$$ Draw a diagram indicating

for which x's this series is:
absolutely convergent, conditionally convergent, and divergent. Indicate your reasoning.

<u>Solution</u>:
We use the ratio test for absolute convergence by letting $u_n = \frac{(x+5)^n}{n} \Rightarrow u_{n+1} = \frac{(x+5)^n(x+5)}{n+1}$

$$\rho = \lim_{n\to\infty} \left| \frac{(x+5)^n(x+5)}{n+1} \cdot \frac{n}{(x+5)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+5)n}{n+1} \right|$$

$$= |x+5| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |x+5|$$

- Converges absolutely when $\rho = |x+5| < 1 \Leftrightarrow$
$$-1 < x+5 < 1 \Leftrightarrow -6 < x < -4$$

- diverges when $\rho = |x+5| > 1 \Leftrightarrow$
$$|x+5| > 1 \Leftrightarrow x > -4 \text{ \& } x < -6$$

<u>Check endpts</u>: $x = -6$ and $x = -4$

$x = -6$, $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges conditionally by AST and the harmonic series.

$x=-4$, $\sum_{n=1}^{\infty} \frac{(x+5)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which

diverges (harmonic series).

So, radius of convergence is 1 and the interval of convergence is $[-6, -4)$