

MARK BOX		
PROBLEM	POINTS	
1 a - r	35	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
%	100	

NAME: Answer Key

SSN: _____

please check the box of your section below

Section 003 (MW 9:05 pm)

or

Section 004 (MW 10:10 pm)

INSTRUCTIONS:

- (1) To receive credit you must:
 - (a) **work in a logical fashion, show all your work, indicate your reasoning;**
no credit will be given for an answer that just appears;
 such explanations help with partial credit
 - (b) when applicable put your answer on/in the line/box provided
 - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.
 Check that your copy of the exam has all of the problems.
- (3) You may **not** use a calculator, books, personal notes.
- (4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- (5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8th ed.):
 Sections 6.6, 6.8, 6.9, 7.1-7.4, 7.6, 7.7, 8.1-8.8, 10.1-10.10. .

Problem Inspiration: homework and old exams.

Solutions will be available on the course homepage later this afternoon.

1. Fill in the blanks (each worth 1 point) and boxes (each worth 2 points).

1a. $\int \frac{dx}{x} = \underline{\ln} |x| + C$

1b. $D_x e^x = \underline{e^x}$

1c. If $a > 0$ but $a \neq 1$, then $D_x a^x = \underline{a^x \ln a}$

Hint: $a^x = e^{\ln(a^x)} = e^{x \ln(a)}$. Your answer should **not** have an "e" in it.

$$\begin{aligned}
 D_x(a^x) &= D_x e^{x \ln a} \\
 &= (e^{x \ln a}) (D_x e^{x \ln a}) \\
 &= (a^x) (\ln a)
 \end{aligned}$$

1d. $D_x \tan x = \underline{\sec^2 x}$

1e. $\int \sec x \tan x = \underline{\sec x} + C$

1f. Integration by parts formula:

$$\int \underline{u} \underline{dv} = \underline{uv} - \int v du$$

1g. Trig substitution: (recall that the *integrand* is the function you are integrating) if the integrand involves $a^2 - u^2$, then one makes the substitution $u = \underline{a \sin \theta}$

1h. Trig substitution: if the integrand involves $a^2 + u^2$, then one makes the substitution $u = \underline{a \tan \theta}$

1i. Partial Fraction Decomposition. If one wants to integrate $\frac{f(x)}{g(x)}$ where f and g are polynomials and $[\text{degree of } f] \geq [\text{degree of } g]$, then one must first do long division

1j. Integral Test: $a_n > 0$

Let $f: [1, \infty) \rightarrow \mathbb{R}$ be so that

- $a_n = f(\underline{n})$ for each $n \in \mathbb{N}$
- f is a continuous function
- f is a positive function
- f is a decreasing function.

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x) dx$ converges.

1k. Comparison Test: $a_n > 0$

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

1l. Limit Comparison Test: $a_n > 0$

Let $b_n > 0$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$.

If 0 $< L < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

1m. Ratio Test: $a_n > 0$

Let $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

- If $\rho < \underline{\quad 1 \quad}$ then $\sum a_n$ converges.
- If $\rho > \underline{\quad 1 \quad}$ then $\sum a_n$ diverges.
- If $\rho = \underline{\quad 1 \quad}$ then the test is inconclusive.

1n. Root Test: $a_n > 0$

Let $\rho = \lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}}$.

- If $\rho < \underline{\quad 1 \quad}$ then $\sum a_n$ converges.
- If $\rho > \underline{\quad 1 \quad}$ then $\sum a_n$ diverges.
- If $\rho = \underline{\quad 1 \quad}$ then the test is inconclusive.

1o. Alternating Series Test: $a_n > 0$

If

- $a_n > a_{n+1}$ for each $n \in \mathbb{N}$
- $\lim_{n \rightarrow \infty} a_n = \underline{0}$

then $\sum (-1)^n a_n$ converges

1p. n^{th} -term test: a_n 's are arbitrary

If $\lim_{n \rightarrow \infty} a_n \neq 0$ or $\lim_{n \rightarrow \infty} a_n$ does not exist, then $\sum a_n$ diverges.

1q. Consider the interval $I = (a - R, a + R)$ center about $x = a$ and of radius R .

Let $y = f(x)$ be a function that can be differentiated N times $x = a$. Then the N^{th} -order Taylor polynomial $y = P_N(x)$ of f about a is (your answer should have a summation sign \sum in it)

$$P_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!} (x-a)^n$$

1r. Consider the interval $I = (a - R, a + R)$ center about $x = a$ and of radius R .

Let $y = f(x)$ be a function that can be differentiated $N + 1$ times for each $x \in I$.

Consider the the N^{th} -order Taylor Reminder term $R_N(x)$, where $f(x) = P_N(x) + R_N(x)$.

Then an upper bound for $|R_N(x)|$ for an $x \in I$ is:

$$|R_N(x)| \leq \left[\max_{x \in (a-R, a+R)} |f^{(N+1)}(x)| \right] \frac{1}{(N+1)!} |x-a|^{N+1}$$

2.

$$D_x (\cos(\ln x)) = [-\sin(\ln x)] \cdot \frac{1}{x}$$

3.

$$D_x 7^{(x^2)} = 7^{(x^2)} (2x) (\ln 7)$$

4.

$$\int (\tan x) (\sec^7 x) dx = \frac{\sec^7 x}{7} + C$$

Remark: box your substitution box for more partial credit.

$$u = \sec x$$

$$du = \sec x \tan x dx$$

$$\int (\tan x) (\sec^7 x) dx = \int \sec^6 x (\sec x \tan x dx)$$

$$= \int u^6 du$$

$$= \frac{u^7}{7} + C$$

$$= \frac{\sec^7 x}{7} + C$$

5.

$$\int x^2 \arctan x \, dx = \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{\ln |1+x^2|}{6} + C$$

Remark: box your substitution box for more partial credit.

parts

$$u = \arctan x \quad dv = x^2 \, dx$$

$$du = \frac{dx}{1+x^2} \quad v = \frac{x^3}{3}$$

Long Division

$$x^2 + 1 \overline{) x^3}$$

$$\underline{x^3 + x}$$

$$-x$$

$$\int x^2 \tan^{-1} x \, dx \stackrel{\text{parts}}{=} \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$$

L.D.

$$\frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int \left[x - \frac{x}{x^2+1} \right] dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \int x \, dx + \frac{1}{3} \frac{1}{2} \int \frac{1}{x^2+1} (2x \, dx)$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3} \frac{x^2}{2} + \frac{1}{6} \ln |x^2+1| + C$$

6.

$$\int \frac{x^2}{\sqrt{4-x^2}} dx = 2 \arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C$$

Remark: box your substitution box for more partial credit.

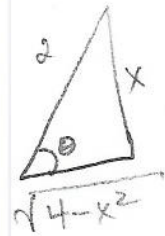
$$a^2 - u^2 \rightarrow u = a \sin \theta$$

$$x = 2 \sin \theta$$

$$\rightarrow \sin \theta = \frac{x}{2}$$

$$dx = 2 \cos \theta d\theta$$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2 \theta} = 2 \cos \theta \rightarrow \cos \theta = \frac{\sqrt{4-x^2}}{2}$$



$$\int \frac{x^2}{\sqrt{4-x^2}} dx = \int \frac{4 \sin^2 \theta \cdot 2 \cos \theta d\theta}{2 \cos \theta} = 4 \int \sin^2 \theta d\theta$$

$$= 4 \cdot \frac{1}{2} \int (1 - \cos 2\theta) d\theta$$

$$= 2\theta - 2 \cdot \frac{\sin 2\theta}{2} + C = 2\theta - 2 \sin \theta \cos \theta + C$$

$$= 2 \arcsin\left(\frac{x}{2}\right) - 2 \left(\frac{x}{2}\right) \left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

7. Let R be the region enclosed by

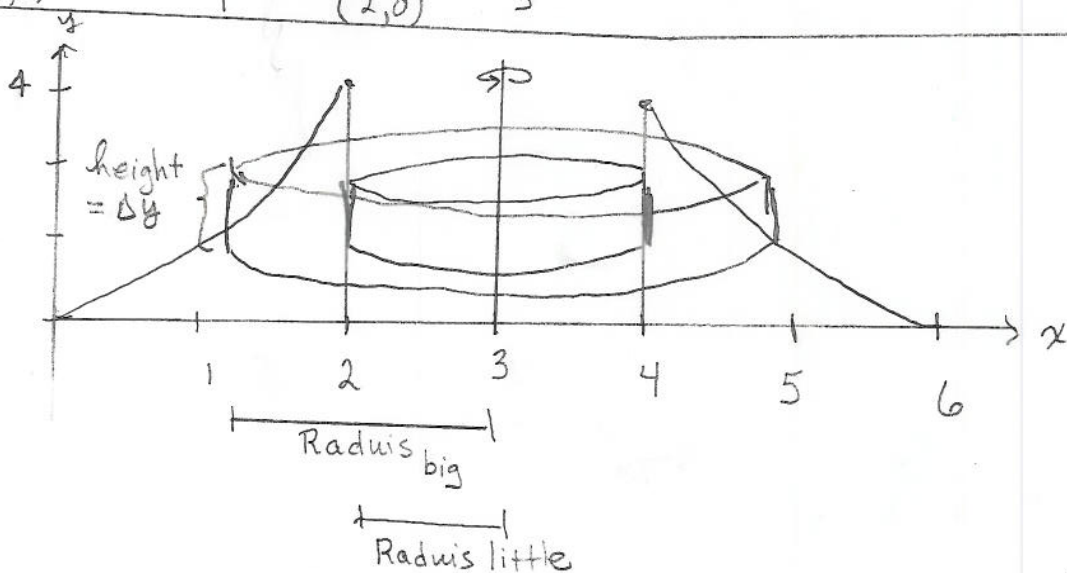
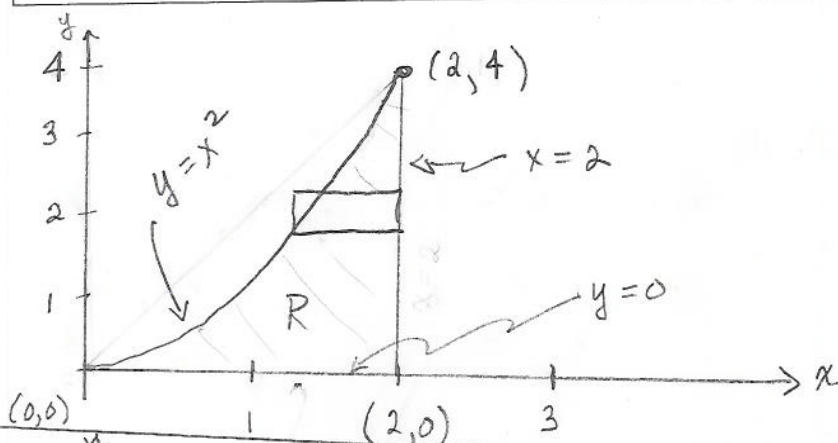
$$y = x^2 \quad \text{and} \quad x = 2 \quad \text{and} \quad y = 0.$$

Let V be the volume of the solid obtained by revolving the region R about the line $x = 3$.

7a. Make a rough sketch below of the region R , labeling the important points.

7b. Using the ~~disk~~/washer method, express the volume V as an integral (or maybe 2 integrals). You do NOT have to evaluate the integral(s).

$$V = \pi \int_{y=0}^{y=4} \left[(3 - \sqrt{y})^2 - 1 \right] dy$$



$$\begin{aligned} \text{Volume of typical element} &= \pi R_{\text{big}}(\text{height}) - \pi R_{\text{little}}(\text{height}) \\ &= \pi (R_{\text{big}} - R_{\text{little}})(\text{height}) = \pi \left[(3 - \sqrt{y})^2 - (3 - 2)^2 \right] \Delta y \end{aligned}$$

8.

$$\lim_{n \rightarrow \infty} \frac{3n^{5/2} + 7n^2 + 9}{-17n^{5/2} + 3n^2 - 9n - 18} = \frac{3}{-17}$$

divide thru by $n^{\text{highest power}} = n^{5/2}$

$$\lim_{n \rightarrow \infty} \frac{3n^{5/2} + 7n^2 + 9}{-17n^{5/2} + 3n^2 - 9n - 18} = \lim_{n \rightarrow \infty} \frac{\frac{3n^{5/2}}{n^{5/2}} + \frac{7n^2}{n^{5/2}} + \frac{9}{n^{5/2}}}{\frac{-17n^{5/2}}{n^{5/2}} + \frac{3n^2}{n^{5/2}} - \frac{9n}{n^{5/2}} - \frac{18}{n^{5/2}}}$$

$$= \lim_{n \rightarrow \infty} \frac{3 + \frac{7}{n^{1/2}} + \frac{9}{n^{5/2}}}{-17 + \frac{3}{n^{1/2}} - \frac{9}{n^{3/2}} - \frac{18}{n^{5/2}}} = \frac{3 + 0 + 0}{-17 + 0 - 0 - 0}$$

9.

$$\lim_{n \rightarrow \infty} \frac{8n^{15} - 7n^{10} + 19}{-5n^{13} + 6n^8 - 6n^5 + 9} = -\infty$$

divide thru by $n^{\text{highest power}} = n^{15}$

$$\lim_{n \rightarrow \infty} \frac{8n^{15} - 7n^{10} + 19}{-5n^{13} + 6n^8 - 6n^5 + 9} = \lim_{n \rightarrow \infty} \frac{8 - \frac{7}{n^5} + \frac{19}{n^{15}}}{-\frac{5}{n^2} + \frac{6}{n^7} - \frac{6}{n^{10}} + \frac{9}{n^{15}}}$$

$$= \frac{8}{0} = \pm \infty \quad (?)$$

$$(?) \quad \frac{8n^{15} - 7n^{10} + 19}{-5n^{13} + 6n^8 - 6n^5 + 9} \stackrel{n \text{ big}}{\approx} \frac{8n^{15}}{-5n^{13}} = -\frac{8}{5}n^2 < 0.$$

On problems 10 and 11b, check the correct box and then indicate your reasoning below.
 A correctly checked box without appropriate explanation will receive no points.

10. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$

absolutely convergent

conditionally convergent

divergent

abs. conv? Consider $\sum \frac{\ln n}{n}$

CT: $\frac{1}{n} \leq \frac{\ln n}{n}$ for n big enough
 \uparrow $\sum \frac{1}{n}$ divg (harmonic series) (p-series, $p=1 \leq 1$)

CT $\Rightarrow \sum \frac{\ln n}{n}$ divg. So not abs. conv.

Cond. conv? try AST so $a_n = \frac{\ln n}{n}$

① $f(x) = \frac{\ln x}{x} \Rightarrow f'(x) = \frac{\frac{1}{2}x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0$ $\swarrow \alpha \geq 3$

so $a_n > a_{n+1}$ for $n \geq 3$

② $\lim_{x \rightarrow \infty} \frac{\ln x}{x} \stackrel{\frac{\infty}{\infty}}{\text{L'H}} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$

AST $\Rightarrow \sum (-1)^n \frac{\ln n}{n}$ conv.

11a Let $a_n = \frac{n^3 (n!)}{(2n)!}$. Find $\frac{a_{n+1}}{a_n}$. Simplify your answer so that no factorial sign (i.e., !) appears.

$$\text{answer: } \frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^3 \frac{n+1}{(2n+1)(2n+2)} \stackrel{\text{or}}{=} \left(\frac{n+1}{n}\right)^3 \frac{1}{2(2n+1)}$$

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^3 ((n+1)!)}{(2(n+1))!} \cdot \frac{(2n)!}{n^3 \cdot n!} = \frac{(n+1)^3}{n^3} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \\ &= \left(\frac{n+1}{n}\right)^3 \frac{n! (n+1)}{n!} \frac{(2n)!}{(2n)! (2n+1)(2n+2)} \end{aligned}$$

11b. $\sum_{n=17}^{\infty} (-1)^n \frac{n^3 (n!)}{(2n)!}$



absolutely convergent



conditionally convergent



divergent

abs conv? Consider $\sum \frac{n^3 n!}{(2n)!}$ & use Ratio Test

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{2(2n+1)}$$

$$= 1^3 \cdot 0 = 0 < 1$$

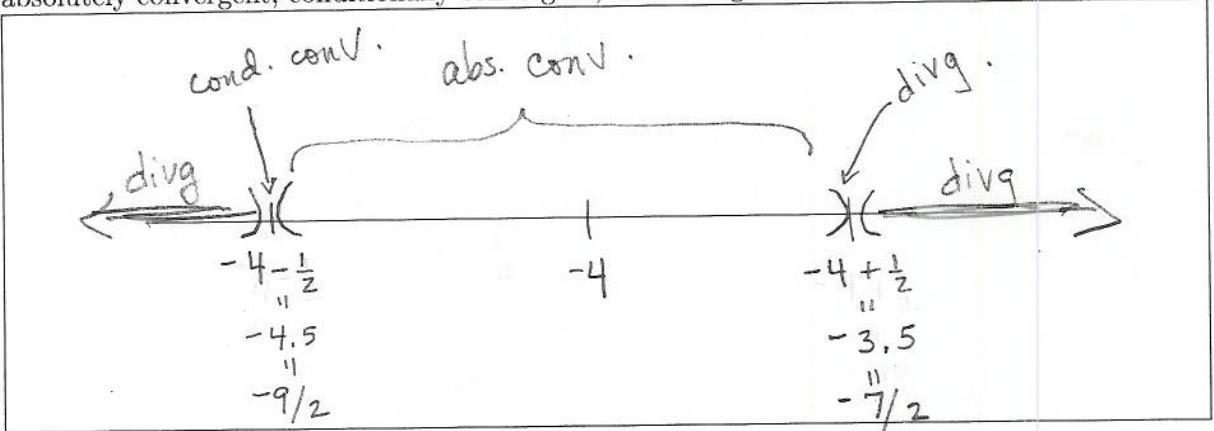
Ratio Test $\Rightarrow \sum \frac{n^3 n!}{(2n)!}$ conv.

12. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(2x+8)^n}{n} = \sum_{n=1}^{\infty} \frac{2^n (x-4)^n}{n}$$

center = -4

As we did in class, in the box below draw a diagram indicating for which x 's this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning.



Ratio Test for abs. conv.

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1} (x+4)^{n+1}}{n+1} \cdot \frac{n}{2^n (x+4)^n} \right| = 2|x+4| \lim_{n \rightarrow \infty} \frac{n}{n+1}$$

$$= 2|x - -4| < 1 \iff |x - -4| < \frac{1}{2}$$

Check endpoints

$$x = -\frac{7}{2} : \sum \frac{(2x+8)^n}{n} = \sum \frac{1}{n} \leftarrow \text{divges, harmonic series or } p\text{-series, } p=1 \leq 1$$

$$x = -\frac{9}{2} : \sum \frac{(2x+8)^n}{n} = \sum \frac{(-1)^n}{n} \leftarrow \text{cond. conv.}$$

- $\sum \frac{1}{n}$ divg
- $\sum \frac{(-1)^n}{n}$ conv. by AST

13. Let

$$f(x) = (1+x)^{3/2}$$

and $a = 0$.

Find the 3rd-order Taylor polynomial of $y = f(x)$ about (or at) $a = 0$.

$P_3(x) =$

$$f(x) = (1+x)^{3/2}$$

$$f(0) = 1$$

$$f'(x) = \frac{3}{2} (1+x)^{1/2}$$

$$f'(0) = \frac{3}{2}$$

$$f^{(2)}(x) = \frac{3}{2} \cdot \frac{1}{2} (1+x)^{-1/2}$$

$$f^{(2)}(0) = \frac{3}{4}$$

$$f^{(3)}(x) = -\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} (1+x)^{-3/2}$$

$$f^{(3)}(0) = -\frac{3}{8}$$

$$P_3(x) = \sum_{n=0}^3 \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\frac{3}{8 \cdot 2 \cdot 2} = \frac{1}{16}$$

$$= 1 + \frac{3}{2}x + \frac{3}{4} \cdot \frac{1}{2!} x^2 + \frac{-3}{8} \cdot \frac{1}{3!} x^3$$
$$= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$$

either one is ok

14. As in problem 13, let

$$f(x) = (1+x)^{3/2}$$

and $a = 0$.

Let $f(x) = P_3(x) + R_3(x)$, where $y = P_3(x)$ is the 3rd-order Taylor polynomial of $y = f(x)$ about $a = 0$ and $y = R_3(x)$ is the corresponding remainder (i.e., error) term.

Consider the interval $I = (-0.5, 0.5)$ center about $a = 0$. Fix an $x \in I$. Find a good upper bound for $|R_3(x)|$.

$$|R_3(x)| \leq \frac{9}{16} \cdot \frac{1}{(0.5)^{5/2}} \cdot \frac{|x|^4}{4!}$$

Remark: you only have to carry out the algebra as far as I indicated in class.

$$f^{(4)}(x) = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} (1+x)^{-5/2} = \frac{9}{2^4} (1+x)^{-5/2} \quad \text{or} \quad \frac{9}{16} (1+x)^{-5/2}$$

$$|R_3(x)| \leq \left[\max_{x \in (-\frac{1}{2}, \frac{1}{2})} |f^{(4)}(x)| \right] \frac{1}{4!} |x-0|^4$$

$$= \left[\max_{x \in (-\frac{1}{2}, \frac{1}{2})} \frac{9}{16} \cdot \frac{1}{|1+x|^{5/2}} \right] \frac{|x|^4}{4!}$$

$$\begin{array}{l} \text{if } -\frac{1}{2} < x < \frac{1}{2} \quad \text{then} \quad \frac{1}{2} < 1+x < \frac{3}{2} \\ \text{so} \quad \frac{1}{2} < |1+x| < \frac{3}{2} \end{array}$$

$$\leq \frac{9}{16} \cdot \frac{1}{(\frac{1}{2})^{5/2}} \cdot \frac{|x|^4}{4!} \quad \text{ok to her}$$

$$= \frac{9}{2^4} \cdot 2^2 \cdot 2^{1/2} \cdot \frac{|x|^4}{4!} = \frac{9\sqrt{2}}{4} \cdot \frac{|x|^4}{4!}$$