

MARK BOX		
PROBLEM	POINTS	
1	36	
2	6	
3	2	
4	8	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
%	100	

NAME: Answer Key

SSN: _____

please check the box of your section below

Section 003 (MW 9:05 pm)

or

Section 004 (MW 10:10 pm)

INSTRUCTIONS:

- (1) To receive credit you must:
 - (a) **work in a logical fashion, show all your work, indicate your reasoning;**
no credit will be given for an answer that *just appears*;
such explanations help with partial credit
 - (b) when applicable put your answer on/in the line/box provided
 - (c) if no such line/box is provided, then box your answer
- (2) The MARK BOX indicates the problems along with their points.
Check that your copy of the exam has all of the problems.
- (3) You may **not** use a calculator, books, personal notes.
- (4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.
- (5) This exam covers (from *Calculus* by Anton, Bivens, Davis 8th ed.):
Sections 10.1 – 10.6 .

Problem Inspiration:

- 1-3. a course handout - you were warned
- 4-6. homework from § 10.1
7. homework from § 10.3
- 8-10. homework from § 10.6

Solutions will be available on the course homepage later this afternoon.

For problems 1, 2, and 3, fill in the blanks.

Hint: I do NOT want to see the words absolute nor conditional on this page!

1. For problem 1, let $\sum a_n$ be a positive-termed series (i.e. $a_n \geq 0$ for each $n \in \mathbb{N}$).

1a. Integral Test Let $f: [1, \infty) \rightarrow \mathbb{R}$ be so that

- $a_n = f(\underline{n})$ for each $n \in \mathbb{N}$
- f is a continuous function
- f is a positive function
- f is a decreasing function.

Then $\sum a_n$ converges if and only if $\int_1^{\infty} f(x) dx$ converges.

1b. Comparison Test

- If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converge, then $\sum a_n$ converge.
- If $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ diverge, then $\sum a_n$ diverge.

1c. Limit Comparison Test Let $b_n > 0$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$. If 0 $< L <$ ∞ , then $\sum a_n$ converges if and only if $\sum b_n$ converges

1d. Ratio Test Let $\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$.

- If $\rho < \underline{1}$ then $\sum a_n$ converges.
- If $\rho > \underline{1}$ then $\sum a_n$ diverges.
- If $\rho = \underline{1}$ then the test is inconclusive.

1e. Root Test Let $\rho = \lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}}$.

- If $\rho < \underline{1}$ then $\sum a_n$ converges.
- If $\rho > \underline{1}$ then $\sum a_n$ diverges.
- If $\rho = \underline{1}$ then the test is inconclusive.

2. For problem 2, we now have an alternating series, i.e., $\sum (-1)^n a_n$ where $a_n > 0$ for each $n \in \mathbb{N}$.

Alternating Series Test: If

- $a_n > a_{n+1}$ for each $n \in \mathbb{N}$
- $\lim_{n \rightarrow \infty} a_n = \underline{0}$

then $\sum (-1)^n a_n$ converges

3. For problem 3, we now have an arbitrary series $\sum a_n$ (some terms might be positive, some might be negative, all might be positive, etc ...).

n^{th} -term test If $\lim_{n \rightarrow \infty} a_n \neq 0$ or $\lim_{n \rightarrow \infty} a_n$ does not exist, then $\sum a_n$ diverges.

4.

$$\lim_{n \rightarrow \infty} \frac{4n^3 + 6n^2 - 17n + 9}{-5n^3 + 7n^2 - 9n - 18} = -\frac{4}{5}$$

$$\frac{4n^3 + 6n^2 - 17n + 9}{-5n^3 + 7n^2 - 9n - 18} = \frac{4 + \frac{6}{n} - \frac{17}{n^2} + \frac{9}{n^3}}{-5 + \frac{7}{n} - \frac{9}{n^2} - \frac{18}{n^3}}$$

$$\xrightarrow{n \rightarrow \infty} \frac{+4 + 0 - 0 + 0}{-5 + 0 - 0 - 0} = -\frac{4}{5}$$

5.

$$\lim_{n \rightarrow \infty} \frac{7n^2 + 9}{-5n + 2} = -\infty$$

Hint: watch your plus and minus.

$$\frac{7n^2 + 9}{-5n + 2} = \frac{7 + \frac{9}{n^2}}{-\frac{5}{n} + \frac{2}{n^2}} \xrightarrow{n \rightarrow \infty} \frac{\cancel{7+0}}{\cancel{0+0}} = \frac{\cancel{7}}{\cancel{0}} = \frac{?}{+} \infty$$

if n is really big, $\frac{7n^2 + 9}{-5n + 2} = \frac{\text{a positive number}}{\text{a negative number}} = \text{a neg. \#}$

6.

$$\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$$

Hint: L'Hopital

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} \xrightarrow{\frac{\infty}{\infty} \text{ L'H}} \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

7. Geometric Series

7a. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

7b. Find the sum of the below series. (Note that the sum begins at $n = 10$ instead of $n = 0$.)

$$\sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-2} = \frac{1}{3^7 \cdot 2}$$

You only have to carry the algebra out as far as I indicated in class.

Notice if $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \dots$$

(7b) $\sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-2}$ \leftarrow geometric series, $r = \frac{1}{3}$, $|r| < 1$, converges

$$\sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-2} = \left(\frac{1}{3}\right)^8 + \left(\frac{1}{3}\right)^9 + \left(\frac{1}{3}\right)^{10} + \dots$$

\uparrow want 1 here so factor out $\left(\frac{1}{3}\right)^8$

$$= \left(\frac{1}{3}\right)^8 \left[1 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots \right]$$

$$= \left(\frac{1}{3}\right)^8 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

$$\stackrel{(7a)}{=} \frac{1}{3^8} \left[\frac{1}{1-\frac{1}{3}} \right] = \frac{1}{3^8} \left[\frac{1}{\frac{2}{3}} \right]$$

$$= \frac{1}{3^8} \cdot \frac{3}{2} = \frac{1}{3^7 \cdot 2}$$

On problems 8 - 10, check the correct box and then indicate your reasoning below. A correctly checked box without appropriate explanation will receive no points.

8. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{e}\right)^n$

absolutely convergent

conditionally convergent

divergent

Hint: $\frac{\pi}{e} \approx \frac{3.14}{2.7} \approx 1.16$.

(a) abs. conv? no!

$$\sum_{n=1}^{\infty} \left| (-1)^n \left(\frac{\pi}{e}\right)^n \right| = \sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^n$$

diverges, geom. series, $r = \frac{\pi}{e}$, $|r| > 1$

(b) does $\sum (-1)^n \left(\frac{\pi}{e}\right)^n$ converge? No

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{\pi}{e}\right)^n = \sum_{n=1}^{\infty} \left(-\frac{\pi}{e}\right)^n$$

geom. series, $r = -\frac{\pi}{e}$, $|r| > 1$, divg.

9. $\sum_{n=17}^{\infty} (-1)^n \frac{1}{n!}$



absolutely convergent



conditionally convergent



divergent

(a) abs conv?

Consider $\sum_{n=17}^{\infty} \left| (-1)^n \frac{1}{n!} \right| = \sum_{n=17}^{\infty} \frac{1}{n!}$

Ratio Test "a_n = n!"

$$\rho = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1}$$

$$= \lim_{n \rightarrow \infty} \frac{n!}{(n!) (n+1)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 < 1$$



Convergence

10. $\sum_{n=2}^{\infty} (-1)^n \frac{n^2}{n^3+8}$

absolutely convergent

conditionally convergent

divergent

(a) abs. conv? No

Consider $\sum_{n=2}^{\infty} |(-1)^n \frac{n^2}{n^3+8}| = \sum_{n=2}^{\infty} \frac{n^2}{n^3+8}$

(bs) $a_n = \frac{n^2}{n^3+8} \stackrel{n \text{ big}}{\approx} \frac{n^2}{n^3} = \frac{1}{n} = b_n$

LCT. $L = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^3+8}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^3}{n^3+8} = 1$
 $0 < 1 < \infty$

$\sum b_n = \sum \frac{1}{n} \Rightarrow$ diverge $\left\{ \begin{array}{l} \cdot p\text{-series, } p=1, p \geq 1 \\ \text{or} \\ \cdot \text{harmonic series} \end{array} \right.$

$\Rightarrow \sum_{n=2}^{\infty} \frac{n^2}{n^3+8}$ diverges

(b) Conditional conv?

Consider $\sum_{n=2}^{\infty} (-1)^n \frac{n^2}{n^3+8}$ & use A.S.T.

• Is $a_n > a_{n+1}$? Yes

$f(x) = \frac{x^2}{x^3+8} \Rightarrow$

$f'(x) = \frac{2x(x^3+8) - x^2(3x^2)}{(x^3+8)^2} = \frac{-x^4 + 16x}{(x^3+8)^2} = \frac{x(16-x^3)}{(x^3+8)^2} < 0$

$f'(x) < 0$ if x is big enough so $a_n > a_{n+1}$ if n is big enough

• $\lim_{n \rightarrow \infty} \frac{n^2}{n^3+8} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1+\frac{8}{n^3}} = \frac{0}{1+0} = 0$

So by AST, $\sum (-1)^n \frac{n^2}{n^3+8}$ converges

$$16 - x^3 < 0$$

$$\iff (16)^{1/3} < x$$