Prof. Girardi | Math 142.001/002 | Spring 2003 | 02.13.03 | Exam 1

MARK BOX		
PROBLEM	POINTS	
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
%	100	

NAME: KEY

SSN: ____________

Section 001 (MW 9:05)

or

Section 002 (MW 10:10)

INSTRUCTIONS:

(1) To receive credit you must:
  (a) work in a logical fashion, show all your work, indicate your reasoning
  (b) when applicable put your answer on/in the line/box provided
  (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes. Give exact answers: for example, write $\ln 2$ instead of .6931, write $\sqrt{2}$ instead of 1.414, write π instead of 3.1415, write $\frac{1}{3}$ instead of 0.3333.

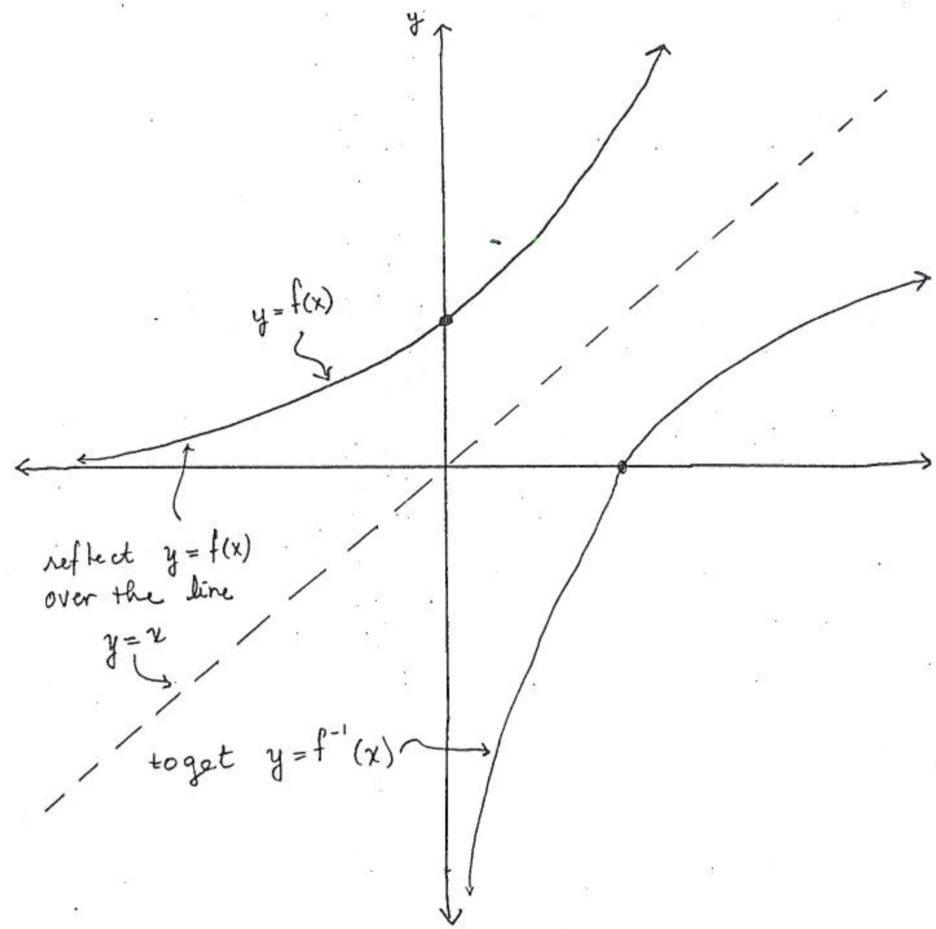
(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Varberg, Purcell, Rigdon, 8th ed.): Chapter 7 (excluding 7.6), Section 8.1, and Section 8.2 .

Problem Inspiration:

1. an example from class
2. homework problem § 7.4 # 5
3. homework problem § 7.1 # 41
4. homework problem § 7.9 # 13
5. homework problem § 7.9 # 23
6. homework problem § 8.1 # 17
7. an example from class
8. Prof. Kustin's Fall 2001 Math 142 Exam 1 # 4
9. an example from class
10. Take Home Quiz (warning: numbers are changed).

1. A function $y = f(x)$ is sketched below.
Sketch the graph of its inverse function, i.e. sketch the graph of $y = f^{-1}(x)$, on the same grid.



2. Solve the equation

$$2\log_9\left(\frac{x}{3}\right) = 1$$

for x. Your answer should **not** have a logarithm nor exponential in it.

ANSWER: $x =$ 9

WAY # 1

$$2 \log_9 \left(\tfrac{x}{3}\right) = 1$$

$$\log_9 \left(\tfrac{x}{3}\right) = \tfrac{1}{2}$$

$$\frac{x}{3} = 9^{1/2}$$

$$\frac{x}{3} = 3$$

$$x = 9$$

WAY # 2

$$2 \log_9 \left(\tfrac{x}{3}\right) = 1$$

$$\log_9 \left(\left(\tfrac{x}{3}\right)^2\right) = 1$$

$$\left(\tfrac{x}{3}\right)^2 = 9^1$$

$$\frac{x}{3} = 3 \quad \leftarrow \text{not } x > 0.$$

$$x = 9$$

3a. For $x > 0$, the natural logarithm of x, i.e. $\ln x$, is **defined** by the definite integral

$$\boxed{\ln x = \int_1^x \frac{dt}{t}}$$

3b. Solve the equation

$$\int_{\frac{1}{3}}^{x} \frac{dt}{t} = 2\int_1^x \frac{dt}{t}$$

for x. Your answer should **not** have a logarithm nor exponential in it.

ANSWER: $x =$ 3

$$\int_{\frac{1}{3}}^{x} \frac{dt}{t} = 2\int_1^x \frac{dt}{t}$$

$$\int_{\frac{1}{3}}^{1} \frac{dt}{t} + \int_1^x \frac{dt}{t} = 2\int_1^x \frac{dt}{t}$$

$$-\int_1^{1/3} \frac{dt}{t} + \int_1^x \frac{dt}{t} = 2\int_1^x \frac{dt}{t}$$

$$-\int_1^{1/3} \frac{dt}{t} = \int_1^x \frac{dt}{t}$$

$$-\ln\left(\tfrac{1}{3}\right) = \ln x$$

$$\ln\left(\left(\tfrac{1}{3}\right)^{-1}\right) = \ln x$$

$$\ln 3 = \ln x$$

$$3 = x$$

4. $D_x\,[\,3\ln(1+e^{5x})\,] = \dfrac{15\,e^{5x}}{1+e^{5x}}$

$$= 3 \cdot \frac{1}{1+e^{5x}} \cdot D_x\,(1+e^{5x})$$

$$= \frac{3}{1+e^{5x}} \cdot \left(0 + e^{5x}\, D_x(5x) \right)$$

$$= \frac{3}{1+e^{5x}} \left(e^{5x}\,(5) \right)$$

5. $D_x\,(x^{1+x}) = x^{1+x}\left(\ln x + \dfrac{1+x}{x}\right) \overset{\text{or}}{=} x^{x}\,(x\ln x + 1 + x)$

$y = x^{1+x}$ & want $\dfrac{dy}{dx}$

$\ln y = \ln\left(x^{1+x}\right)$

$\ln y = (1+x)(\ln x)$

=> differentiate both sides with respect to x.

$$\frac{1}{y}\frac{dy}{dx} = [D_x(1+x)]\ln x + (1+x)\,D_x(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + \frac{(1+x)}{x}$$

$$\frac{dy}{dx} = y\left(\ln x + \frac{1+x}{x}\right)$$

$$= x^{1+x}\left(\ln x + \frac{1+x}{x}\right)$$

$$= x \cdot x^{x}\left(\frac{x\ln x + 1 + x}{x}\right)$$

$$= x^{x}\,(x\ln x + 1 + x)$$

6.

$$\boxed{\int \frac{3x^2+2x}{x+1}\,dx = \frac{3x^2}{2} - x + \ln|x+1| + C}$$

$$\begin{array}{r|l} & 3x - 1 \\ \hline x+1 & 3x^2 + 2x + 0 \\ & -(3x^2+3x) \\ \hline & -x + 0 \\ & -(-x-1) \\ \hline & 1 \end{array}$$

$$\int \frac{3x^2+2x}{x+1}\,dx = \int \left[3x - 1 + \frac{1}{x+1} \right] dx$$

$$= \frac{3x^2}{2} - x + \int \frac{dx}{x+1}$$

$$\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}}$$

$$= \frac{3x^2}{2} - x + \int \frac{du}{u}$$

$$= \frac{3x^2}{2} - x + \ln|u| + C$$

$$= \frac{3x^2}{2} - x + \ln|x+1| + C$$

7.

$$\boxed{\int \sin^4(17x)\cos^3(17x)\,dx = \frac{1}{17}\left(\frac{\sin^5(17x)}{5} - \frac{\sin^7(17x)}{7} \right) + C}$$

$$\cancel{\boxed{\begin{array}{l} u = \cos(17x) \\ du = -17\sin(17x)\,dx \end{array}}} \quad \text{or} \quad \boxed{\begin{array}{l} v = \sin(17x) \\ dv = 17\cos(17x)\,dx \end{array}}$$

$$\int \sin^4(17x)\cos^3(17x)\,dx =$$

$$= \frac{1}{17}\int \sin^4(17x)\cos^2(17x)\,[17\cos(17x)\,dx]$$

$$= \frac{1}{17}\int \sin^4(17x)\,(1-\sin^2(17x))\,[17\cos(17x)\,dx]$$

$$= \frac{1}{17}\int v^4(1-v^2)\,dv$$

$$= \frac{1}{17}\int (v^4 - v^6)\,dv$$

$$= \frac{1}{17}\left(\frac{v^5}{5} - \frac{v^7}{7} \right) + C$$

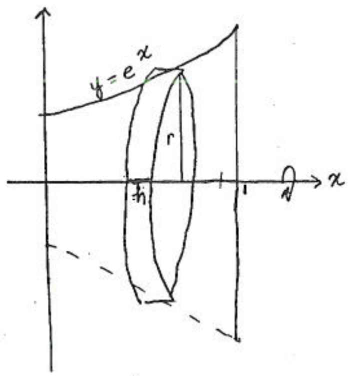
$$= \frac{1}{17}\left(\frac{\sin^5(17x)}{5} - \frac{\sin^7(17x)}{7} \right) + C$$

$$\underline{\underline{\text{or}}} \quad \frac{\sin^5(17x)}{85} - \frac{\sin^7(17x)}{119} + C$$

8. The volume of the solid generated by revolving the region bounded by:

$y = e^x$ and the x-axis and the y-axis and $x = 1$

about the x-axis is $\boxed{\dfrac{\pi\,(e^2 - 1)}{2}}$.



Volume of a typical element

$= (\text{area of base})(\text{height})$

$= (\pi r^2)\,(h)$

$= \pi\,(e^x)^2\ \Delta x$

$= \pi e^{2x}\ \Delta x$

$$\text{Volume} = \int_0^1 \pi e^{2x}\,dx$$

$$= \pi \int_{x=0}^{x=1} e^{2x}\,dx$$

$$\boxed{\begin{array}{ll} u = 2x & x = 0 \Rightarrow u = 0 \\ du = 2\,dx & x = 1 \Rightarrow u = 2 \end{array}}$$

$$= \frac{\pi}{2} \int_{u=0}^{u=2} e^u\,du = \frac{\pi}{2}\, e^u \Big|_{u=0}^{u=2}.$$

$$= \frac{\pi}{2}\,(e^2 - e^0)$$

9. The solution to the differential equation

$$\frac{dy}{dt} = 12\,y(t)$$

subject to the condition that $y(2) = 7$ is $\boxed{y(t) = \dfrac{7e^{12t}}{e^{24}}}$.

$$\frac{dy}{dt} = 12\,y(t)$$

$$\frac{1}{y(t)}\frac{dy}{dt} = 12$$

$$\parallel$$

$$D_t\left(\ln y(t)\right) = D_t(12t)$$

$$\ln y(t) = 12t + C$$

$$e^{\ln y(t)} = e^{12t + C}$$

$$y(t) = e^c\, e^{12t}$$

$$7 = y(2) = e^c\, e^{12\cdot 2}$$

$$7 = e^c \cdot e^{24}$$

$$e^c = \frac{7}{e^{24}}$$

19. (1) The rate of decay of a radioactive substance is proportional to the amount of such substance present.
Today we have P_0 grams of a radioactive substance.
Given that one-fourth of the substance decays every 7 years (2), how much will be left t years from today? *Clearly explain your notation.*

ANSWER: $P_0\, e^{\frac{1}{7}\ln\left(\frac{3}{4}\right)t} \overset{\text{or}}{=} P_0\left(\frac{3}{4}\right)^{t/7}$ grams

Let t_0 = initial time = today

t = the # of years after t_0

$P(t)$ = the # of grams of the radioactive substance at time t.

$P_0 = P(0)$

(1) $\Rightarrow \dfrac{dP}{dt} = k\,P(t) \Rightarrow P(t) \overset{(3)}{=} P_0\, e^{kt}$

Use (2) to find k:

$\frac{3}{4}P_0 \overset{(2)}{=} P(7) \overset{(3)}{=} P_0\, e^{k(7)}$

$\Rightarrow P_0\, e^{7k} = \frac{3}{4}P_0$

$\Rightarrow e^{7k} = \frac{3}{4}$

$\Rightarrow \ln e^{7k} = \ln\left(\frac{3}{4}\right)$

$\Rightarrow 7k = \ln\left(\frac{3}{4}\right)$

$\Rightarrow k = \frac{1}{7}\ln\left(\frac{3}{4}\right)$

$$P(t) = P_0\, e^{\frac{1}{7}\ln\left(\frac{3}{4}\right)t}$$
$$= P_0\left[e^{\frac{1}{7}\ln\left(\frac{3}{4}\right)}\right]^t$$
$$= P_0\left[e^{\ln\left(\left(\frac{3}{4}\right)^{1/7}\right)}\right]^t$$
$$= P_0\left[\left(\frac{3}{4}\right)^{1/7}\right]^t$$
$$= P_0\left(\frac{3}{4}\right)^{t/7}$$