Prof. Girardi	Math 142	Fall 2003	12.08.03	Final Exam

MARK BOX		
PROBLEM	POINTS	
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
15	5	
16	5	
17	5	
18	5	
19	5	
20	5	
%	100	

NAME: Key

SSN: ____________

Section 007 (MW 12:20 pm)

or

Section 008 (MW 2:30)

INSTRUCTIONS:

(1) To receive credit you must:
  (a) work in a logical fashion, show all your work, indicate your reasoning
  (b) justify your reasoning
  (c) when applicable put your answer on/in the line/box provided
  (d) if no such line/box is provided, then box your answer.

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes. Give exact answers: for example, write $\ln 2$ instead of .6931, write $\sqrt{2}$ instead of 1.414, write π instead of 3.1415, write $\frac{1}{3}$ instead of 0.3333.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Varberg, Purcell, Rigdon, 8th ed.): Chs 7-10, Sections 11.1, 12.6-12.8 .

1.

$$D_x\left[17^{3x^2+1}\right] = (\ln 17)\ 17^{3x^2+1}\ (6x)$$

$$D_x\, 17^{3x^2+1} = D_x\, e^{\ln 17^{3x^2+1}}$$

$$= D_x\, e^{(3x^2+1)(\ln 17)}$$

$$= e^{(3x^2+1)(\ln 17)} \cdot (6x)(\ln 17)$$

$$= e^{\ln 17^{3x^2+1}}\ (6x)(\ln 17)$$

$$= 17^{3x^2+1}\ (6x)(\ln 17)$$

2.

$$D_x\,[\ln(\cos(4x))] = \frac{-4\sin(4x)}{\cos(4x)} = -4\tan(4x)$$

3.

$$D_x\left((1+x)^{2x}\right) = (1+x)^{2x}\left[2\ln(1+x) + \frac{2x}{1+x}\right]$$

$$y = (1+x)^{2x}$$

$$\ln y = 2x \ln(1+x)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\ln(1+x) + \frac{2x}{1+x}$$

4. The rate of decay of a radioactive substance is proportional to the amount of such substance present. Today we have 30 grams of a radioactive substance.
Given that one-fourth of the substance decays every 7 years, how much will be left t years from today? *Clearly explain your notation.*

ANSWER: $P(t) = 30\, e^{\frac{1}{7}\ln(3/4)\, t}$ grams

HINT: your answer should have a t in it.

$$P(t) = P_0 e^{kt}$$

$$P(t) = 30 e^{kt}$$

$$P(7) = \frac{3}{4} P_0$$

$$30 e^{7k} = \frac{3}{4} \cdot 30$$

$$e^{7k} = \frac{3}{4}$$

$$7k = \ln\left(\frac{3}{4}\right)$$

$$k = \frac{1}{7} \ln\left(\frac{3}{4}\right)$$

5.

$$\int \frac{dx}{\sqrt{x}\ (1+x)} = 2 \tan^{-1}(x^{1/2}) + C$$

$$u = x^{1/2}$$

$$u^2 = x$$

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)} = 2\int \frac{du}{1+u^2} = 2\tan^{-1} u + C$$

6.

$$\int \sec^3 x \tan^3 x \, dx = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

$$\boxed{\begin{aligned} u &= \sec x \\ du &= \sec x \tan x \, dx \end{aligned}}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan^2 x \, (\sec x \tan x \, dx)$$

$$= \int \sec^2 x \, (\sec^2 x - 1) \, (\sec x \tan x \, dx)$$

$$= \int u^2 (u^2 - 1) \, du$$

$$= \int (u^4 - u^2) \, du$$

7.

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

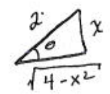
$$\int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + c$$

8.

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} \qquad +C$$

$x = 2\sin\theta \rightarrow \sin\theta = \frac{x}{2} \rightarrow$



$\cos^2\theta + \sin^2\theta = 1$

$$\int \frac{x^2\,dx}{\sqrt{4-x^2}} = \int \frac{4\sin^2\theta\,(2\cos\theta\,d\theta)}{\sqrt{4-4\sin^2\theta}}$$

$$= \frac{4 \cdot 2}{2} \int \frac{\sin^2\theta\,\cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}}$$

$$= 4\int \sin^2\theta\,d\theta$$

$$\overset{*7}{=} 4\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right] + C$$

$$= 2\theta - \sin(2\theta) + C$$

$$= 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\arcsin\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

9.

$$\int x^2 \arctan x\,dx = \frac{x^3\arctan x}{3} - \frac{x^2}{6} + \frac{1}{6}\ln|x^2+1| \qquad +C$$

$u = \arctan x \qquad dv = x^2\,dx$

$du = \frac{dx}{1+x^2} \qquad v = \frac{x^3}{3}$

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & \underline{x^3 + x} \\ & -x \end{array}$$

$$\int x^2 \arctan x\,dx = \frac{x^3}{3}\arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\left[\int \left(x - \frac{x}{x^2+1}\right)dx\right]$$

$$= \frac{x^3\arctan x}{3} - \frac{1}{3}\int x\,dx + \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= \frac{x^3\arctan x}{3} - \frac{1}{3}\,\frac{x^2}{2} + \frac{1}{6}\ln|x^2+1| + C$$

10. $\displaystyle\int \frac{x^4+2x+2}{x^5+x^4}\,dx = \frac{2x^{-3}}{-3} + \ln|x+1| + C$

$$\frac{x^4+2x+2}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

$$x^4+2x+2 = Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4$$

$x=0 \Rightarrow \boxed{2=D}$ $\qquad x=-1 \Rightarrow \boxed{1=E}$

$x^4 : \; 1 = A+E \longrightarrow \boxed{A=0}$

$x^3 : \; 0 = A+B \longrightarrow \boxed{B=0}$

$x^2 : \; 0 = B+C \longrightarrow \boxed{C=0}$

$x^1 : \; 2 = C+D$

Const : $2 = D$

$$\int \frac{x^4+2x+2}{x^3+x^4}\,dx = \int \left(2x^{-4} + \frac{1}{x+1}\right)dx$$

11. $\displaystyle\int \frac{x}{x^4+4x^2+8}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{x^2+2}{2}\right) + C$

$$\int \frac{x\,dx}{x^4+4x^2+8} = \int \frac{x\,dx}{(x^2+2)^2+4}$$

$\boxed{\begin{array}{l} u = x^2+2 \\ du = 2x\,dx \end{array}}$

$$= \frac{1}{2}\int \frac{du}{u^2+2^2}$$

$\boxed{u = 2\tan\theta}$

$$= \frac{1}{2}\int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta+4}$$

$$= \frac{1}{2}\int \frac{2\sec^2\theta}{4\sec^2\theta}\,d\theta$$

$$= \frac{1}{2}\,\frac{1}{2}\int d\theta$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{u}{2}\right)$$

12.

$$\int e^x \cos x\,dx = \frac{e^x(\cos x + \sin x)}{2} + C$$

$u = \cos x \qquad dv = e^x\,dx$

$du = -\sin x \qquad v = e^x$

$u = \sin x \qquad dv = e^x\,dx$

$du = \cos x\,dx \qquad v = e^x$

$$\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$$

$$\Rightarrow 2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$$

13. Let c be some constant with $c \neq 0$.

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x = e^c$$

$$L = \lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$$

$$\Rightarrow \ln L = \lim_{x\to\infty} \ln\left(1+\frac{c}{x}\right)^x$$

$$= \lim_{x\to\infty} \frac{\ln\left(\frac{x+c}{x}\right)}{\frac{1}{x}}$$

$$\overset{L'H}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{x}{x+c}\cdot c\left(\frac{d}{dx}\frac{1}{x}\right)}{\left(\frac{d}{dx}\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{cx}{x+c}$$

$$= \lim_{x\to\infty} \frac{c}{1+\frac{c}{x}} = c$$

$$\ln L = c \Rightarrow L = e^c$$

14. $\int_0^\infty \frac{dx}{x+1} = \infty$ or DNE

$$\int_0^\infty \frac{dx}{x+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x+1}$$

$$= \lim_{b\to\infty} \ln|x+1| \Big|_{x=0}^{x=b}$$

$$= \lim_{b\to\infty} \ln|b+1| - \cancel{\ln 1} \to 0$$

15. $\lim_{n\to\infty} \frac{12n^{17}+188n^7-19n}{4n^{16}-n^9+10} = \infty$

$$\lim_{n\to\infty} \frac{12n^{17}+188n^7-19n}{4n^{16}-n^9+10} = \lim_{n\to\infty} \frac{12+\frac{188}{n^{10}}-\frac{19}{n^{16}}}{\frac{4}{n}-\frac{1}{n^8}+\frac{10}{n^{17}}}$$

16. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n^2-1}}$

- ☐ absolutely convergent
- ☒ conditionally convergent
- ☐ divergent

abs. conv? Consider $\sum \frac{1}{\sqrt{n^2-1}}$

LCT. $a_n = \frac{1}{\sqrt{n^2-1}} \approx \frac{1}{\sqrt{n^2}} = \frac{1}{n} = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2-1}} = \sqrt{1} = 1 \quad 0 < \ < \infty$$

$\sum \frac{1}{n}$ diverges (p-series, $p=1$) $\Rightarrow \sum \frac{1}{\sqrt{n^2-1}}$ divg.

cond. conv. $\sum (-1)^n \frac{1}{\sqrt{n^2-1}} \Rightarrow a_n = \frac{1}{\sqrt{n^2-1}}$

A.S.T. $\cdot \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$

$\cdot \frac{1}{\sqrt{n^2-1}} > \frac{1}{\sqrt{(n+1)^2-1}} = a_n > a_{n+1}$

So $\sum (-1)^n \frac{1}{\sqrt{n^2-1}}$ converges.

17. $\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)!}{(2n)!}$

- ☒ absolutely convergent
- ☐ conditionally convergent
- ☐ divergent

abs. conv? $a_n = \frac{(n+1)!}{(2n)!}$

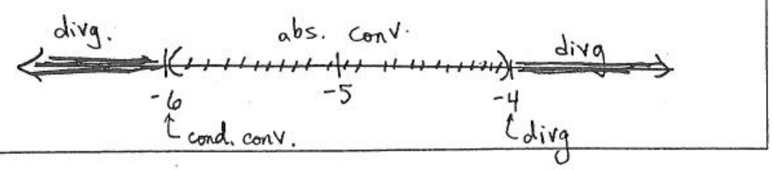
Ratio Test

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)!}{(n+1)!} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \frac{n+2}{(2n+1)(2n+2)} = 0 < 1$$

$\Rightarrow \sum \frac{(n+1)!}{(2n)!}$ converges.

18. Consider the formal power series
$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n}.$$
As we did in class, in the box below draw a diagram indicating for which x's this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning.



$$\rho = \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{n+1} \cdot \frac{n}{(x+5)^n} \right| = \lim_{n\to\infty} |x+5| \frac{n}{n+1} = |x+5| < 1$$

Check endpts:

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n} \quad \underline{\underline{x=-4}} \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{divg, harmonic series}$$

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n} \quad \underline{\underline{x=-6}} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{cond. conv by} \quad \& \text{ AST}$$

19. Consider the function $f(x) = e^{2x}$. Let $a = 0$. We know that we can write the function as
$$f(x) = P_3(x) + R_3(x)$$
where P_3 is the third order Taylor polynominal of f about $a = 0$ and R_3 is the corresponding remainder term.

19a Find $P_3(x)$.

$P_3(x) = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 = 1 + 2x + 2x^2 + \frac{4x^3}{3}$

$f(x) = e^{2x}$	$f(0) = 1$
$f^{(1)}(x) = 2e^{2x}$	$f^{(1)}(0) = 2$
$f^{(2)}(x) = 4e^{2x}$	$f^{(2)}(0) = 4$
$f^{(3)}(x) = 8e^{2x}$	$f^{(3)}(0) = 8$
$f^{(4)}(x) = 16e^{2x}$	$f^{4}(c) = 16e^{2c}$

19b Find a formula for the remainder term $R_3(x)$. Your answer should have a "c" in it. **Be sure to indicate where c lies.**

$R_3(x) = \frac{16e^{2c}}{4!}x^4 \;\underline{\underline{\text{or}}}\; \frac{2}{3}e^{2c}x^4$

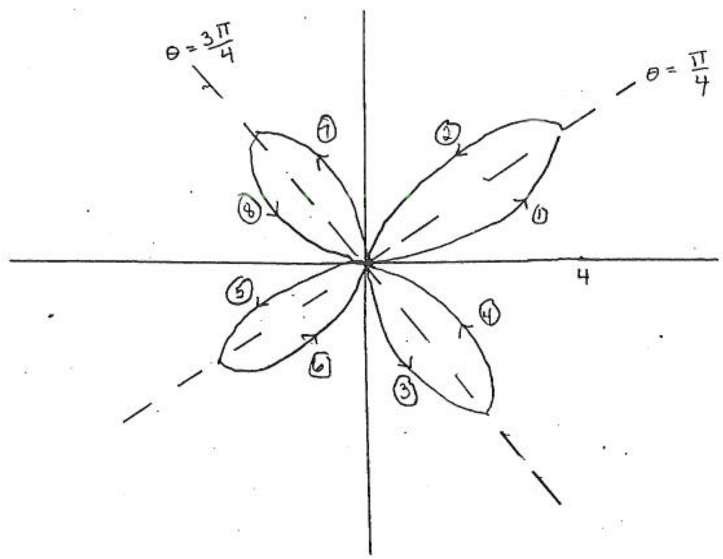
19c Find a good upper bound for $|R_3(x)|$ for $-0.5 \le x \le 0.5$. Your answer should **not** have a "c" in it but you do not have to do arithmetic. **Do you work on the back of the previous page.**

$|R_3(x)| \le \frac{2}{3}e^{2(0.5)}(0.5)^4 = \frac{2e}{3\,2^4} = \frac{e}{3 \cdot 2^3} = \frac{e}{24}$

20. Consider the polar equation $r = 4\sin(2\theta)$.

20a. Using the method from class, sketch the graph of this polar equation.

Make your chart on the back of the previous page.



20b. Express the area enclosed by this polar equation as an integral (but you do not have to evaluate this integral). Use symmetry.

Area = $8 \cdot \frac{1}{2} \int_0^{\pi/4} \left(4\sin(2\theta)\right)^2 d\theta = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (4\sin(2\theta))^2 d\theta$