Prof. Girardi | Math 142 | Fall 2003 | 11.19.03 | Exam 3

| MARK BOX | | |
|---|---|---|
| PROBLEM | POINTS | |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| USCvsClemson | 2 | |
| % | 100 | |

NAME: Key

SSN: ____________

**Section 007 (MW 12:20pm)**

or

**Section 008 (MW 2:30pm)**

**INSTRUCTIONS:**

(1) To receive credit you must:
  (a) work in a logical fashion, show all your work, indicate your reasoning
  (b) when applicable put your answer on/in the line/box provided
  (c) if no such line/box is provided, then box your answer

(2) The MARK BOX indicates the problems along with their points. Check that your copy of the exam has all of the problems.

(3) You may **not** use a calculator, books, personal notes. Give exact answers: for example, write $\ln 2$ instead of .6931, write $\sqrt{2}$ instead of 1.414, write $\pi$ instead of 3.1415, write $\frac{1}{3}$ instead of 0.3333.

(4) During this exam, do not leave your seat. If you have a question, raise your hand. When you finish: turn your exam over, put your pencil down, and raise your hand.

(5) This exam covers (from *Calculus* by Varberg, Purcell, Rigdon, 8th ed.): Chapter 10 and Section 11.1 .

**ADDITIONAL INSTRUCTIONS**

**In problems 1 - 3, find the limit of each sequence.**
**In problems 5 - 8, check the appropriate box of what that series is doing.**
**Always justify your answer.**



**1.**

$$\lim_{n\to\infty} \frac{17n^2 - 3n + 1}{3n^2 - 17n + 100} = \frac{17}{3}$$

Way # 1

$$\lim_{n\to\infty} \frac{\boxed{17}n^2 - 3n + 1}{\boxed{3}n^2 - 17n + 100} = \frac{17}{3}$$

degree of top = degree of bottom so we get

WAY # 2

$$\lim_{n\to\infty} \frac{17n^2 - 3n+1}{3n^2 - 17n+100} = \lim_{n\to\infty} \frac{17 - \frac{3}{n} + \frac{1}{n^2}}{3 - \frac{17}{n} + \frac{100}{n^2}} = \frac{17-0+0}{3-0+0}$$

$$= \frac{17}{3}$$

**2.**

$$\lim_{n\to\infty} \frac{3n^2+4}{n-1} = \infty \quad \underline{\text{or}} \quad \text{diverges} \quad \text{or} \quad \text{DNE}$$

**Warning: beware, the degree of the polynomial in the numerator does not equal the degree of the polynomial in the denominator. So divide through by the dominating power to see what happens towards infinity.**

$$\lim_{n\to\infty} \frac{3n^2+4}{n-1} = \lim_{n\to\infty} \frac{3+\frac{4}{n^2}}{\frac{1}{n}-\frac{1}{n^2}} = \cancel{\frac{3+0}{0-0}} = \cancel{\frac{3}{0}}$$



**3.**

$$\lim_{n\to\infty} (0.999999999)^n = 0$$

**Warning: do not confuse a sequence with a series.**

- Recall

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & |r| < 1 \\ \text{DNE} & |r| > 1 \\ \text{DNE} & r = -1 \quad (\text{oscillating}) \\ 1 & r = 1 \quad (r^n = 1 \text{ for each } n \end{cases}$$

here $r = 0.999999999$

so $|r| < 1$.

4. Explicitly find the sum of the geometric series

$$\sum_{n=4}^{\infty} 8\left(\frac{1}{6}\right)^n = \boxed{\frac{1}{135}}.$$

**Note that the above series starts at $n = 4$.** Recall that if $|r| < 1$ then (starting at $n = 0$)

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}. \qquad r = \frac{1}{6}, \ |r| < 1.$$

$$\sum_{n=4}^{\infty} 8\left(\tfrac{1}{6}\right)^n = 8\left[\left(\tfrac{1}{6}\right)^4 + \left(\tfrac{1}{6}\right)^5 + \left(\tfrac{1}{6}\right)^6 + \left(\tfrac{1}{6}\right)^7 + \ldots\right]$$

want 1 here so factor out $\left(\frac{1}{6}\right)^4$

$$= 8\left(\tfrac{1}{6}\right)^4 \left[1 + \left(\tfrac{1}{6}\right)^1 + \left(\tfrac{1}{6}\right)^2 + \left(\tfrac{1}{6}\right)^3 + \ldots\right]$$

$$= \boxed{\frac{8}{6^4} \quad \frac{1}{1-\frac{1}{6}}}$$

ok to here but you could keep going

$$\overset{\text{or}}{=} \frac{8}{6^4} \cdot \frac{6}{5} = \frac{8}{6^3} \cdot \frac{1}{5}$$

$$= \frac{2^3}{2^3 \cdot 3^3} \cdot \frac{1}{5} = \frac{1}{3^3 \cdot 5} = \frac{1}{27 \cdot 5}$$

$$\begin{array}{r} \overset{3}{27} \\ \times\ 5 \\ \hline 135 \end{array}$$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!}$

☒ absolutely convergent

☐ conditionally convergent

☐ divergent

① Absolutely convergent? Well, consider

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{3^n}{n!} \right| = \sum_{n=1}^{\infty} \frac{3^n}{n!}$$

factorial $\Rightarrow$ Ratio Test with $a_n = \frac{3^n}{n!}$

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n \to \infty} \frac{3}{n+1} = 0 < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{3^n}{n!} \text{ converges}$$

6. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n+1}}$

☐ absolutely convergent

☒ conditionally convergent

☐ divergent

① Abs. conv.? Consider $\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{\sqrt{n+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

LCT $\quad a_n = \frac{1}{\sqrt{n+1}} \underset{\text{big}}{\overset{n}{\approx}} \frac{1}{\sqrt{n}} \equiv b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{\sqrt{n+1}}{1} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{1/2} = 1^{1/2} = 1$$

$0 < 1 < \infty$

So $\sum \frac{1}{\sqrt{n+1}}$ & $\sum \frac{1}{\sqrt{n}}$ do the same thing.

$\sum \frac{1}{\sqrt{n}}$ diverges ( p-series $p = \frac{1}{2} \leq 1$)

So $\sum \frac{1}{\sqrt{n+1}}$ diverge. So $\sum \frac{(-1)^n}{\sqrt{n+1}}$ is not abs. conv.

② Conditionally conv? Have $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n = \frac{1}{\sqrt{n+1}}$.

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$
- $\frac{1}{\sqrt{n+1}} > \frac{1}{\sqrt{(n+1)+1}} \Rightarrow a_n > a_{n+1}$

A.S.T. $\Rightarrow \sum (-1)^n \frac{1}{\sqrt{n+1}}$ converges



7. $\sum_{n=1}^{\infty} (-1)^n \frac{(n^2+1)}{2n^2+n-1}$

☐ absolutely convergent

☐ conditionally convergent

☒ divergent by $n^{th}$ - term test for divergence.

**Think before you dive in.**

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+n-1} = \frac{1}{2} \neq 0$$

$$\lim_{n\to\infty} (-1)^n \frac{n^2+1}{2n^2+n-1} \quad \text{DNE.} \neq 0$$

oscillating

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n 3^n}{n^n}$

- [x] absolutely convergent
- [ ] conditionally convergent
- [ ] divergent

**Hint: Root Test.**

① abs. conv?

Consider $\sum \left| (-1)^n \frac{2^n 3^n}{n^n} \right| = \sum \frac{2^n \cdot 3^n}{n^n}$

Root Test

$$\lim_{n\to\infty} \left[ \frac{2^n \cdot 3^n}{n^n} \right]^{1/n} = \lim_{n\to\infty} \frac{2\cdot 3}{n} = 0 < 1.$$

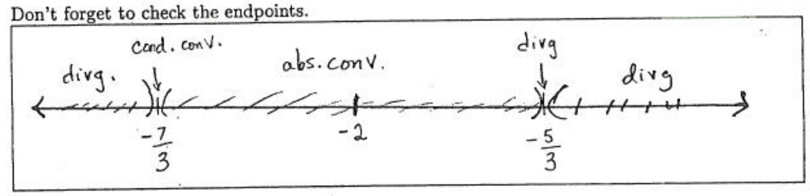
Root Test $\Rightarrow \sum \frac{2^n \cdot 3^n}{n^n}$ converges.



9. Consider the formal power series

$$\sum_{n=1}^{\infty} \frac{(3x+6)^n}{n}.$$

As we did in class, in the box below draw a diagram indicating for which $x$'s this series is: absolutely convergent, conditionally convergent, and divergent. Of course, indicate your reasoning. Don't forget to check the endpoints.




$$\lim_{n\to\infty} \left| \frac{(3x+6)^{n+1}}{n+1} \cdot \frac{n}{(3x+6)^n} \right| = \lim_{n\to\infty} |3x+6| \frac{n}{n+1}$$

$= |3x+6| = 3|x - -2| < 1 \iff |x - (-2)| < \frac{1}{3}$

center

endpts: $-2 \pm \frac{1}{3}$ → $-5/3$, $-7/3$

Check endpts

- $\displaystyle\sum_{n=1}^{\infty} \frac{(3x+6)^n}{n} \overset{x=-5/3}{=\!=} \sum_{n=1}^{\infty} \frac{(+1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, diverges harmonic series (or p-series, $p = 1 \le 1$)

- $\displaystyle\sum_{n=1}^{\infty} \frac{(3x+6)^n}{n} \overset{x=-7/3}{=\!=} \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← not abs. conv. by

but $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n = \frac{1}{n}$

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$
- $\frac{1}{n} > \frac{1}{n+1} \Rightarrow a_n > a_{n+1}$

$\xrightarrow{AST} \sum \frac{(-1)^n}{n}$ conv.

**10.** Consider the function $f(x) = \ln(2+x)$. Let $a = 0$. We know that we can write the function as

$$f(x) = P_2(x) + R_2(x)$$

where $P_2$ is the second order Taylor polynominal of $f$ about $a = 0$ and $R_2$ is the corresponding remainder term.

**10a** Find $P_2(x)$.

$P_2(x) = \ln 2 + \frac{1}{2} x^1 - \frac{1}{4}\frac{1}{2!} x^2$

| | |
|---|---|
| $f(x) = \ln(2+x)$ | $f(0) = \ln 2$ |
| $f^{(1)}(x) = (2+x)^{-1}$ | $f^{(1)}(0) = 2^{-1} = \frac{1}{2}$ |
| $f^{(2)}(x) = -(2+x)^{-2}$ | $f^{(2)}(0) = -2^{-2} = -\frac{1}{4}$ |
| $f^{(3)}(x) = 2(2+x)^{-3}$ | $f^{(3)}(c) = 2(2+c)^{-3}$ |

**10b** Find a formula for the remainder term $R_2(x)$. Your answer should have a "$c$" in it, be sure to indicate where $c$ lies.

$R_2(x) = \frac{f^{(3)}(c)}{3!} x^3 = \frac{2}{(2+c)^3} \frac{x^3}{3!}$ where c is between $x$ and 0

**10c** Find a good upper bound for $|R_2(0.5)|$. Your answer should not have a "$c$" in it but you do not have to do arithmetic. **Do you work on the back of the previous page.** OK TO HERE

$|R_2(0.5)| \leq \frac{2}{(2+c)^3} \frac{(0.5)^3}{3!} \leq \boxed{\frac{2}{2^3} \frac{(0.5)^3}{3!}} \overset{\text{or}}{=} \frac{2}{2^3} \frac{1}{2^3} \frac{1}{3!} = \frac{1}{3!\,2^5}$

$0 \leq c \leq 0.5 \Rightarrow 2 \leq 2+c \leq 2.5$