Homework 2. Find the equation $y = p_1(x)$ of the tangent line to the function $f(x) = \frac{1}{x}$ at the point $x_0 = 2$. Express your answer in the form $p_1(x) = d + m(x-2)$ for some constants d & m.

Soln: $p_1(x) = \boxed{\frac{1}{2} + \frac{-1}{4}(x-2)}$.

We have already had on the handout that the equation $y = p_1(x)$ of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0) \ . \qquad (1)$$

Helpful Table for Homework 2		
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(2)$
0	$f^{(0)}(x) \overset{\text{def}}{=} f(x) = x^{-1}$	$f^{(0)}(2) = \frac{1}{2}$
1	$f^{(1)}(x) \overset{\text{def}}{=} f'(x) = -\ x^{-2}$	$f^{(1)}(2) = -\frac{1}{4}$

Using Helpful Table for Homework 2 and the equation (1), we get:

$$\boxed{p_1(x) = \frac{1}{2} + \frac{-1}{4}(x-2) \ .} \qquad (2)$$

Note that (2) is the the reqested form $p_1(x) = d + m(x-2)$ where the constants $d = \frac{1}{2}$ and $m = \frac{-1}{4}$ so WE ARE DONE.

Homework 4. Find the second order Taylor polynomial $y = p_2(x)$ for $f(x) = \frac{1}{x}$ at $x_0 = -2$. First fill in the Helpful Table for Homework 4. Then express your answer in the form

$$p_2(x) = c_0 + c_1(x -^- 2) + c_2(x -^- 2)^2 \quad \text{or} \quad p_2(x) = c_0 + c_1(x+2) + c_2(x+2)^2$$

for some constants c_0, c_1, c_2.

We have from the handout that the second order Taylor polynomial of $y = f(x)$ at x_0 is

$$p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \ . \qquad (3)$$

Helpful Table for Homework 4			
n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(-2)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(-2)}{n!}$
0	$f^{(0)}(x) \overset{\text{def}}{=} f(x) = x^{-1}$	$f(-2) = \frac{-1}{2}$	$\frac{-1}{2}\ \frac{1}{0!} = \frac{-1}{2}$
1	$f^{(1)}(x) \overset{\text{def}}{=} f'(x) = -\ x^{-2}$	$f'(-2) = \frac{-1}{4}$	$\frac{-1}{4}\ \frac{1}{1!} = \frac{-1}{4}$
2	$f^{(2)}(x) \overset{\text{def}}{=} f''(x) = 2x^{-3}$	$f''(-2) = \frac{-2}{8}$	$\frac{-2}{8}\ \frac{1}{2!} = \frac{-1}{8}$

Using equation (3) with the Helpful Table, we get

Soln: $p_2(x) = \boxed{\frac{-1}{2} + \frac{-1}{4}(x+2) + \frac{-1}{8}(x+2)^2}$.

Homework 6. For the function $f(x) = \sin(3x)$ from Example 5, find the Maclaurin polynomials:

$$y = p_1(x),\ y = p_3(x),\ y = p_5(x),\ y = p_7(x),\ y = p_9(x),\ y = p_{11}(x), \text{ and } y = p_{13}(x)\ .$$

First fill out the Helpful Table and then indicate the Maclaurin polynomials in the Solution Table.
We are looking for patterns so you may leave/express, e.g., 3^5 as just 3^5 rather than 243 and 5! as just 5! rather than 120; in short, you do not need a calculator.

The N^{th}-order Taylor polynomial for $y = f(x)$ centered at x_0 is, as motivated on the handout,

$$p_N(x) \;=\; \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n \;=\; \sum_{n=0}^{N} c_n (x-x_0)^n \quad \text{where} \quad c_n = \frac{f^{(n)}(x_0)}{n!}.$$

The <u>N^{th}-*order Maclaurin polynomial*</u> for $y = f(x)$, which is just the N^{th}-order Taylor polynomial for $y = f(x)$ centered at $x_0 = 0$, is therefore

$$p_N(x) \;=\; \sum_{n=0}^{N} c_n x^n \quad \text{where} \quad c_n = \frac{f^{(n)}(0)}{n!}\ . \tag{4}$$

Helpful Table for Homework 6

n	$f^{(n)}(x)$	$f^{(n)}(x_0) \overset{\text{here}}{=} f^{(n)}(0)$	$c_n \overset{\text{def}}{=} \frac{f^{(n)}(x_0)}{n!} \overset{\text{here}}{=} \frac{f^{(n)}(0)}{n!}$
0	$\sin(3x) \overset{\text{note}}{=} +3^0 \sin(3x)$	0	0
1	$3\cos(3x) \overset{\text{note}}{=} +3^1 \cos(3x)$	$+3^1$	$+\frac{3^1}{1!}$
2	$-3^2 \sin(3x)$	0	0
3	$-3^3 \cos(3x)$	-3^3	$-\frac{3^3}{3!}$
4	$+3^4 \sin(3x)$	0	0
5	$+3^5 \cos(3x)$	$+3^5$	$+\frac{3^5}{5!}$
6	$-3^6 \sin(3x)$	0	0
7	$-3^7 \cos(3x)$	-3^7	$-\frac{3^7}{7!}$
8	$+3^8 \sin(3x)$	0	0
9	$+3^9 \cos(3x)$	$+3^9$	$+\frac{3^9}{9!}$
10	$-3^{10} \sin(3x)$	0	0
11	$-3^{11} \cos(3x)$	-3^{11}	$-\frac{3^{11}}{(11)!}$
12	$+3^{12} \sin(3x)$	0	0
13	$+3^{13} \cos(3x)$	$+3^{13}$	$+\frac{3^{13}}{(13)!}$

Using equation (4) with the Helpful Table, we get:

Solution Table for Homework 6

n	$y = p_n(x)$
1	$p_1(x) = \frac{3^1}{1!}x^1$
3	$p_3(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3$
5	$p_5(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5$
7	$p_7(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7$
9	$p_9(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \frac{3^9}{9!}x^9$
11	$p_{11}(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \frac{3^9}{9!}x^9 - \frac{3^{11}}{(11)!}x^{11}$
13	$p_{13}(x) = \frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3 + \frac{3^5}{5!}x^5 - \frac{3^7}{7!}x^7 + \frac{3^9}{9!}x^9 - \frac{3^{11}}{(11)!}x^{11} + \frac{3^{13}}{(13)!}x^{13}$

Bonus problem. In Homework 6, what is the 4^{th}-order Maclaurin polynomial?

Soln: $p_4(x) = \boxed{\frac{3^1}{1!}x^1 - \frac{3^3}{3!}x^3}$.

Helpful Thoughts

To be read before next class but was not be be handed in.

Just to think about. Take another look at Homework 6. Do you notice any pattern in the Taylor coefficients? Why did we only use odd-order Taylor polynomials?

Let's look at Table ?? for a pattern. Notice the pattern for the Maclaurin coefficients c_n's, i.e. for $\{c_n\}_{n=0}^{13}$, or better yet, for $\{c_n\}_{n=0}^{\infty}$? It looks as if

$$c_n \text{ is } \begin{cases} 0 & \text{if } n \text{ is even} \\ +\frac{3^n}{n!} \quad \text{or} \quad -\frac{3^n}{n!} & \text{if } n \text{ is odd} . \end{cases} \tag{5}$$

We just need to figure out the $\pm$ in the case that n is odd. What is causing the pattern? Well, it's the sinus function. To help find the pattern, let's look at an related easier example, $g(x) = \sin x$.

But first, some handy notation useful for splitting up subsets of integers into the (disjoint) union of the even and odd ones. Recall that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and that $\biguplus$ denotes disjoint union.

$$\begin{aligned}
\mathbb{N} &\overset{\text{def}}{=} \{1,2,3,4,5,6,\ldots\} = \text{the natual numbers} &\quad \text{and} \quad& \mathbb{N}_0 \overset{\text{def}}{=} \{0,1,2,3,4,5,\ldots\} \\
2\mathbb{N} &\overset{\text{def}}{=} \{2,4,6,8,10,12,\ldots\} = \text{the even } \mathbb{N}\text{'s} &\quad \text{and} \quad& 2\mathbb{N}_0 \overset{\text{def}}{=} \{0,2,4,6,8,10,\ldots\} \\
2\mathbb{N}-1 &\overset{\text{def}}{=} \{1,3,5,7,9,11,\ldots\} = \text{the odd } \mathbb{N}\text{'s} &\quad \text{and} \quad& 2\mathbb{N}_0+1 \overset{\text{def}}{=} \{1,3,5,7,9,11,\ldots\} \\
\mathbb{N} &\overset{\text{so}}{=} (2\mathbb{N}) \biguplus (2\mathbb{N}-1) &\quad \text{and} \quad& \mathbb{N}_0 \overset{\text{so}}{=} (2\mathbb{N}_0) \biguplus (2\mathbb{N}_0+1)
\end{aligned}$$

There are other (obvious) notations.[1]

Now back to pattern searching with $g(x) = \sin x$. Recall, we want to spit $\mathbb{N}_0$ into $(2\mathbb{N}_0) \biguplus (2\mathbb{N}_0 + 1)$.

to split n into even and odds				without k		get k involved	
	$2\mathbb{N}_0$	$2\mathbb{N}_0+1$	$\mathbb{N}_0$				
k	$2k$	$2k+1$	n	$g^{(n)}(x)$	$g^{(n)}(0)$	$g^{(n)}(x)$	$g^{(n)}(0)$
0	0		0	$\sin x$	0	$(-1)^k \sin x$	0
0		1	1	$\cos x$	1	$(-1)^k \cos x$	$(-1)^k$
1	2		2	$-\sin x$	0	$(-1)^k \sin x$	0
1		3	3	$-\cos x$	-1	$(-1)^k \cos x$	$(-1)^k$
notice a recycling pattern kicking in							
2	4		4	$\sin x \overset{\textbf{notice}}{=} g^{(0)}(x)$	0	$(-1)^k \sin x$	0
2		5	5	$\cos x \overset{\textbf{notice}}{=} g^{(1)}(x)$	1	$(-1)^k \cos x$	$(-1)^k$
3	6		6	$-\sin x \overset{\textbf{notice}}{=} g^{(2)}(x)$	0	$(-1)^k \sin x$	0
3		7	7	$-\cos x \overset{\textbf{notice}}{=} g^{(3)}(x)$	-1	$(-1)^k \cos x$	$(-1)^k$
the recycling pattern starts over again							

TABLE 1

[1] To see if you are understanding, convince yourself that $\mathbb{N} = (3\mathbb{N}) \biguplus (3\mathbb{N}-1) \biguplus (3\mathbb{N}-2)$.

We now see, from the recyling pattern and the the help of the split with the k's, that for $n \in \mathbb{N}_0$,

$$g^{(n)}(0) \;=\; \begin{cases} 0 & \text{if } n \in \mathbb{N}_0 \text{ is even} \\ (-1)^k & \text{is } n \in \mathbb{N}_0 \text{ is odd and of the form } n = 2k+1 \text{ for some } k \in \mathbb{N}_0 \end{cases} .$$

Note, this completely (i.e., covers all cases) gives $\{g^{(n)}(0)\}_{n=0}^{\infty}$.

Now back to the originial problem of pattern searching to find the Maclaurin coefficients $\{c_n\}_{n=0}^{\infty}$ for $f(x) = \sin(3x)$. We start by making a table similar to, and using ideas learned from, Table 1 for $g(x) = \sin x$.

to split n into even and odds				without k		get k involved	
	$2\mathbb{N}_0$	$2\mathbb{N}_0+1$	$\mathbb{N}_0$				
k	$2k$	$2k+1$	n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)$	$c_n = \frac{f^{(n)}(0)}{n!}$
0	0		0	$\sin(3x)$	0	0	0
0		1	1	$3\cos(3x)$	3	$(-1)^k 3$	$(-1)^k \frac{3^1}{1!}$
1	2		2	$-3^2\sin(3x)$	0	0	0
1		3	3	$-3^3\cos(3x)$	-3^3	$(-1)^k 3^3$	$(-1)^k \frac{3^3}{3!}$
notice a recycling pattern kicking in							
2	4		4	$3^4\sin(3x) \overset{\textbf{notice}}{=} 3^4 f^{(0)}(x)$	0	0	0
2		5	5	$3^5\cos(3x) \overset{\textbf{notice}}{=} 3^4 f^{(1)}(x)$	3^5	$(-1)^k 3^5$	$(-1)^k \frac{3^5}{5!}$
3	6		6	$-3^6\sin(3x) \overset{\textbf{notice}}{=} 3^4 f^{(2)}(x)$	0	0	0
3		7	7	$-3^7\cos(3x) \overset{\textbf{notice}}{=} 3^4 f^{(3)}(x)$	-3^7	$(-1)^k 3^7$	$(-1)^k \frac{3^7}{7!}$
the recycling pattern starts over again							

At last, we see (and understand why it really works) the pattern. For each $k \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$,

$$c_{2k} \overset{(*)}{=} 0 \qquad \text{and} \qquad c_{2k+1} \overset{(**)}{=} (-1)^k \frac{3^{2k+1}}{(2k+1)!} .$$

$$p_{2N+1}(x) \overset{\text{by (4)}}{=} \sum_{n=0}^{2N+1} c_n x^n = \sum_{k=0}^{N} c_{2k} x^{2k} + \sum_{k=0}^{N} c_{2k+1} x^{2k+1} \overset{\text{by } (*)}{=} \sum_{k=0}^{N} c_{2k+1} x^{2k+1}$$

$$\overset{\text{by } (**)}{=} \sum_{k=0}^{N} (-1)^k \frac{3^{2k+1}}{(2k+1)!} x^{2k+1} \overset{\text{let } m = k+1}{=} \sum_{m=1}^{N+1} (-1)^{m-1} \frac{3^{2m-1}}{(2m-1)!} x^{2m-1} .$$

Since $p_{2k}(x) \overset{\text{by (4)}}{=} p_{2k-1}(x) + c_{2k}x^{2k} \overset{\text{by } (*)}{=} p_{2k-1}(x)$ for $k \geq 1$, we only considered odd Taylor polynomials.