In this assignment you are given a function $y = f(z)$, where $f\colon (0,\infty) \to \mathbb{R}$, and you are asked to find the limit of $y = f(z)$ as $z \to \infty$, i.e.,

$$\lim_{z\to\infty} f(z) \qquad \text{which can also be written as} \qquad \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} f(z)\,.$$

The limits in this assignment are used often in Calc. II and can be done using just Calc. I knowledge. Recall: the handout *Indeterminate Forms (L'Hôpital's Rule)*, which is a summary/review of L'H rule, can be found on the course homepage under Handouts (see **Preparation**).

Instructions.
First compute the limits by-hand, showing all your work below the box and then putting your answer in the box. **Next** use Maple to check your answers. **Hand in** only your by-hand work.

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1. $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{\ln z}{z} = \boxed{0}$ Hint: L'Hôpital's Rule.

$$\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{\ln z}{z} \overset{\frac{\infty}{\infty}}{\underset{\text{L'H}}{=}} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{D_z \ln z}{D_z z} \overset{\text{Ⓒ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{\frac{1}{z}}{1} \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{1}{z} = 0.$$

2. $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} z^{1/z} = \boxed{1}$ Recall the continuous function $\exp\colon \mathbb{R} \to (0,\infty)$ is given by $\exp(x) = e^x$.

Hint: $\displaystyle\lim_{z\to\infty} z^{1/z}$ has the indeterminate form ∞^0 so consider $\ln\left(z^{1/z}\right) \overset{\text{i.e.}}{=} \left(\frac{1}{z}\right)\ln z \overset{\text{i.e.}}{=} \frac{\ln z}{z}$, and then apply L'H.

First consider $\ln\left(z^{1/z}\right)$. Compute $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \ln\left(z^{1/z}\right) \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \left(\frac{1}{z}\right)\ln z \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{\ln z}{z} \overset{\text{L'H}}{\underset{\text{see } \mathbf{1.}}{=}} 0.$

So $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} z^{1/z} \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \exp\left(\ln\left(z^{1/z}\right)\right) \overset{\text{since the exp function}}{\underset{\text{is continuous}}{=}} \exp\left(\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \ln\left(z^{1/z}\right)\right) = \exp(0) = 1.$ Here is another way to think about the last step. We first computed

$$\ln\left(z^{1/z}\right) \xrightarrow[z\in\mathbb{R}]{z\to\infty} 0$$

and so, since the exp function is continuous,

$$z^{1/z} \overset{\text{Ⓐ}}{=} e^{\ln\left(z^{1/z}\right)} \xrightarrow[z\in\mathbb{R}]{z\to\infty} e^0 = 1.$$

3. Let $c > 0$ be a positive constant. $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} c^{1/z} = \boxed{1}$

Hint: $\ln\left(c^{1/z}\right) = \left(\frac{1}{z}\right)\ln c = \frac{\ln c}{z}$ and since c is a constant, $\ln c$ is also a constant.

Consider $\ln\left(c^{1/z}\right)$. Compute $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \ln\left(c^{1/z}\right) \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \left(\frac{1}{z}\right)\ln c \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \frac{\ln c}{z} \overset{\text{since } \ln c}{\underset{\text{is a constant}}{=}} (\ln c) \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \ln\frac{1}{z} \overset{\text{Ⓒ}}{=}$ $(\ln c)\,0 = 0$. So $\displaystyle\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} c^{1/z} \overset{\text{Ⓐ}}{=} \lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \exp\left(\ln\left(c^{1/z}\right)\right) \overset{\text{since the exp function}}{\underset{\text{is continuous}}{=}} \exp\left(\lim_{\substack{z\to\infty \\ z\in\mathbb{R}}} \ln\left(c^{1/z}\right)\right) = \exp(0) = 1.$

4. Let c be a constant. Think of c as a fixed number, for example $\frac{1}{17}$.

4a. If $0 < c < 1$ then $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} c^z = \boxed{0}$

Hint: $\ln(c^z) = z\ln c$ and since $0 < c < 1$ is a constant, $\ln c$ is also a constant and $\ln c < 0$.

First consider $\ln c^z$. Compute $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \ln c^z \overset{Ⓐ}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} z\ln c \underset{\ln c \text{ is a constant}}{\overset{\text{since}}{=}} (\ln c)\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} z \underset{\ln c<0}{\overset{\text{since}}{=}} -\infty$. Since $\ln c^z \xrightarrow[z\in\mathbb{R}]{z\to\infty} -\infty$ and the exp function is continuous, $c^z \overset{Ⓐ}{=} e^{\ln(c^z)} \xrightarrow[z\in\mathbb{R}]{z\to\infty} \cancel{e^{-\infty} =}\ 0$.

4b. If $c = 1$ then $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} c^z = \boxed{1}$ Hint: For each $z\in\mathbb{R}$ we have $c^z = 1^z = 1$.

$\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} 1^z \underset{1^z=1}{\overset{\text{since}}{=}} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} 1 = 1$.

4c. If $c > 1$ then $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} c^z = \boxed{\infty \text{ DNE is also ok}}$

Hint: $\ln(c^z) = z\ln c$ and since $c > 1$ is a constant, $\ln c$ is also a constant and $\ln c > 0$.

First consider $\ln c^z$. Compute $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \ln c^z \overset{Ⓐ}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} z\ln c \underset{\ln c \text{ is a constant}}{\overset{\text{since}}{=}} (\ln c)\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} z \underset{\ln c>0}{\overset{\text{since}}{=}} \infty$. Since $\ln c^z \xrightarrow[z\in\mathbb{R}]{z\to\infty} \infty$ and the exp function is continuous, $c^z \overset{Ⓐ}{=} e^{\ln(c^z)} \xrightarrow[z\in\mathbb{R}]{z\to\infty} \cancel{e^{\infty} =}\ \infty$.

5. Let c be a constant. $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \left(1+\frac{c}{z}\right)^z = \boxed{e^c}$

Hint: $\lim\limits_{z\to\infty}\left(1+\frac{c}{z}\right)^z$ has the indeterminate form 1^∞ so consider $\ln\left(\left(1+\frac{c}{z}\right)^z\right)$ and rewrite

$\ln\left(\left(1+\frac{c}{z}\right)^z\right) = z\ln\left(1+\frac{c}{z}\right) = \frac{\ln\left(1+\frac{c}{z}\right)}{\frac{1}{z}} \overset{\text{also}}{=} \frac{\ln\left(\frac{z+c}{z}\right)}{\frac{1}{z}}$. Now apply L'H. If needed, use&connect another sheet of paper.

Solution To find $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \left(1+\frac{c}{z}\right)^z$, observe that since $\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \left(1+\frac{c}{z}\right)^z$ is of the indeterminate form 1^∞, as the L'Hopital handout suggests, we want to first consider $\ln\left(\left(1+\frac{c}{z}\right)^z\right)$ since this will allow us to *bring the exponent down* and then rewrite using algebraic manipulations so as to be able to apply L'H's rule. Whenever we do this, we just have to remember to exponentiate our result at the end.

$$\lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \ln\left(\left(1+\frac{c}{z}\right)^z\right) \overset{Ⓐ}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} z\ln\left(1+\frac{c}{z}\right) \underset{Ⓐ}{\overset{\cancel{\infty\cdot 0}}{=}} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{\ln\left(1+\frac{c}{z}\right)}{\frac{1}{z}}$$

$$\underset{\text{L'H}}{\overset{\cancel{\frac{0}{0}}}{=}} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{D_z\ln\left(1+\frac{c}{z}\right)}{D_z\left(\frac{1}{z}\right)} \overset{\text{i.e.}}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{D_z\ln\left(\frac{z+c}{z}\right)}{D_z\left(\frac{1}{z}\right)}$$

$$\overset{ⓒ}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{\left[\frac{z}{z+c}\right] D_z\left(1+\frac{c}{z}\right)}{D_z\left(\frac{1}{z}\right)} \overset{ⓒ}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{\left[\frac{z}{z+c}\right] c\,D_z\left(\frac{1}{z}\right)}{D_z\left(\frac{1}{z}\right)}$$

$$\overset{\text{CH}}{=} \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{cz}{z+c} = \lim\limits_{\substack{z\to\infty\\ z\in\mathbb{R}}} \frac{c}{1} = c.$$

Since $\ln\left(\left(1+\frac{c}{z}\right)^z\right) \xrightarrow[z\in\mathbb{R}]{z\to\infty} c$ and the exp function is continuous, $\left(1+\frac{c}{z}\right)^z \overset{Ⓐ}{=} \exp\left(\ln\left(\left(1+\frac{c}{z}\right)^z\right)\right) \xrightarrow[z\in\mathbb{R}]{z\to\infty} e^c$.