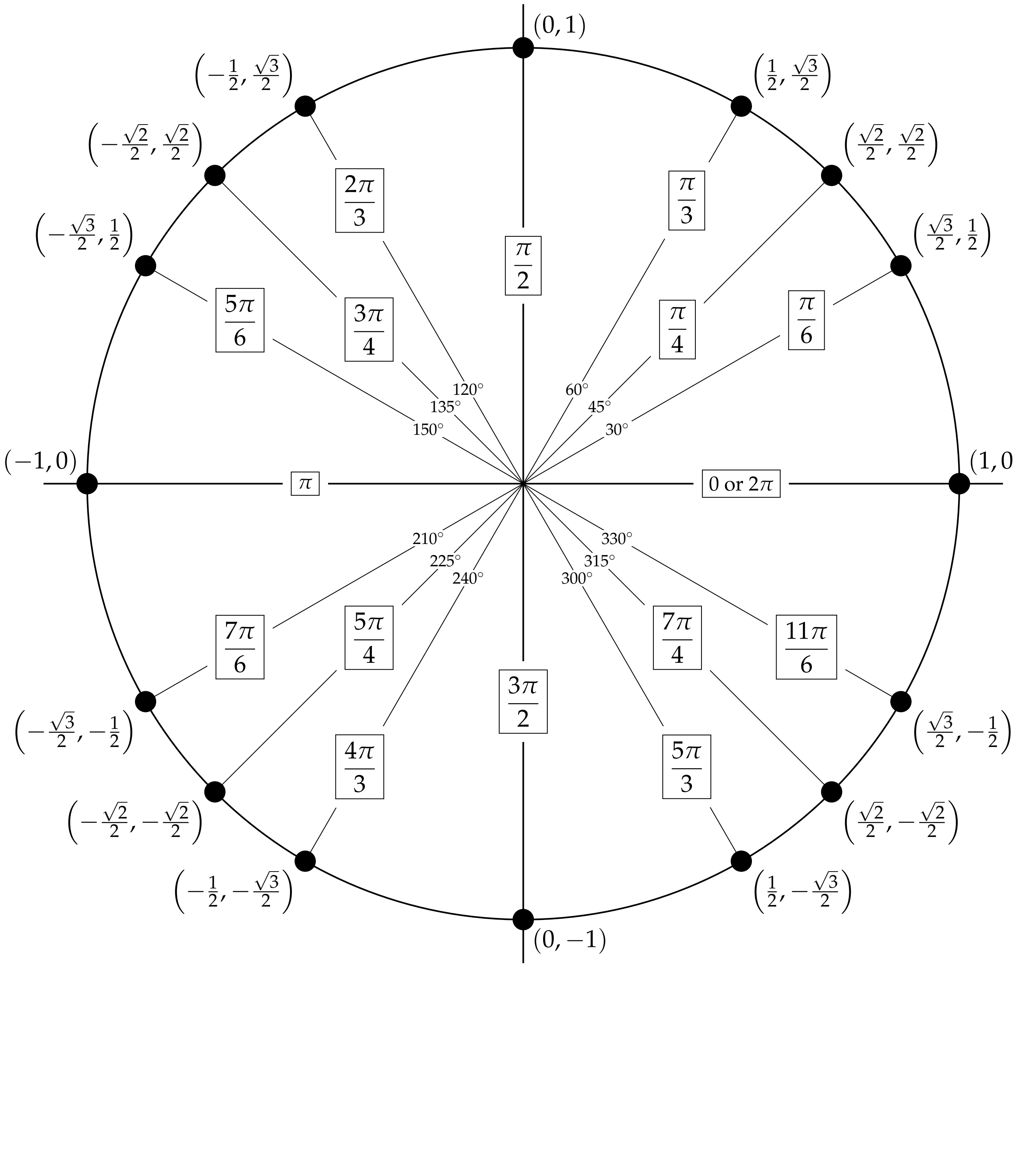
1. Complete the Unit Circle started below.

1.1. Fill in the boxes with the angle measurement in radians, between 0 and 2π, for each of the 16 angles which are measured in degrees.

1.2. Next to each of the 16 points drawn, indicate the (x, y) coordinate of the point on the unit circle.



Instructions For Remaining Trig. Problems 2–5:
First show all your work below the box then put answer in the box.
No credit will be given for an answer just put in a box without proper justification.
Work in a logical fashion, explaining how you arrived at your boxed answer.

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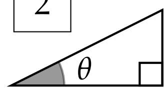
2. Fill in the boxes. You might want to first review the range of the inverse trigonometry functions.

2.1. A reference triangle for $\tan\theta = \frac{\sqrt{3}}{1}$ is:



2.2. Express $\arctan\sqrt{3}$ in radians. ANSWER: $\arctan\sqrt{3} = \boxed{\frac{\pi}{3}}$

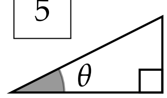
$\left[\arctan\sqrt{3} = \theta\right] \iff \left[\sqrt{3} = \tan\theta \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right] \iff \left[\frac{\sqrt{3}/2}{1/2} = \frac{\sin\theta}{\cos\theta} \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right]$
And since $\tan\theta > 0$, we can also say $0 < \theta < \frac{\pi}{2}$. Now just look at our Unit Circle.

2.3. Express $\arctan\left({}^{-}\sqrt{3}\right)$ in radians. ANSWER: $\arctan\left({}^{-}\sqrt{3}\right) = \boxed{-\frac{\pi}{3}}$

$\left[\arctan\left(-\sqrt{3}\right) = \theta\right] \iff \left[-\sqrt{3} = \tan\theta \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right] \iff \left[-\frac{\sqrt{3}/2}{1/2} = \frac{\sin\theta}{\cos\theta} \text{ and } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right]$
And since $\tan\theta < 0$, we can also say $-\frac{\pi}{2} < \theta < 0$. Now just look at our Unit Circle.
Note $\frac{5\pi}{3}$ is incorrect since $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$.

3. Fill in the boxes or circle the correct answer.

3.0. A reference triangle for $\tan\theta = \frac{4}{3}$ is



3.1. Can θ be between 0 and $\frac{\pi}{2}$? (1^{st} quadrant) circle one: **[YES]** or NO

Answer if (and only if) you circled YES. Then $\sin\theta = \boxed{\frac{4}{5}}$

3.2. Can θ be between $\frac{\pi}{2}$ and π? (2^{nd} quadrant) circle one: YES or **[NO]**

Answer if (and only if) you circled YES. Then $\sin\theta = \boxed{}$

3.3. Can θ be between π and $\frac{3\pi}{2}$? (3^{rd} quadrant) circle one: **[YES]** or NO

Answer if (and only if) you circled YES. Then $\sin\theta = \boxed{-\frac{4}{5}}$

3.4. Can θ be between $\frac{3\pi}{2}$ and 2π? (4^{th} quadrant) circle one: YES or **[NO]**

Answer if (and only if) you circled YES. Then $\sin\theta = \boxed{}$

4. Let $x = 5\sec\theta$ and $0 < \theta < \frac{\pi}{2}$.
Without using inverse trigonometric functions, express $\tan\theta$ as a function of x.

ANSWER: $\tan\theta = \boxed{\dfrac{\sqrt{x^2-25}}{5}}$

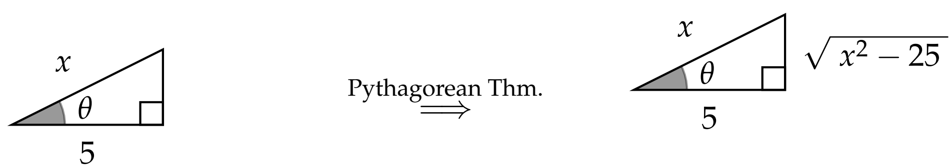
4&5 Way #1.

Techniques in this way are need in the upcoming section on Trig. Substitution.
Note

$$x = 5\sec\theta \implies \frac{x}{5} = \sec\theta \overset{\text{note}}{=} \frac{1}{\cos\theta}\,.$$

So if $0 < \theta < \frac{\pi}{2}$, then from the below reference triangles we infer

$$\frac{x}{5} = \frac{\text{hyp}}{\text{adj}} \implies \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-25}}{5}\,.$$



So $\tan\theta = \pm\frac{\sqrt{x^2-25}}{5}$ and we just now need to determine whether to take the plus or the minus:

$$\tan\theta = \begin{cases} +\frac{\sqrt{x^2-25}}{5} & \text{if } \tan\theta \geq 0\text{, which is the case for } 0 < \theta < \frac{\pi}{2} \\ -\frac{\sqrt{x^2-25}}{5} & \text{if } \tan\theta \leq 0\text{, which is the case for } \frac{\pi}{2} < \theta < \pi\,. \end{cases} \tag{1}$$

5. Let $x = 5\sec\theta$ and $\frac{\pi}{2} < \theta < \pi$.
Without using inverse trigonometric functions, express $\tan\theta$ as a function of x.

ANSWER: $\tan\theta = \boxed{-\dfrac{\sqrt{x^2-25}}{5}}$

4&5 Way #2.

Techniques in this way are need in the upcoming section on Trig. Integration.
Recall that

$$\cos^2\theta + \sin^2\theta = 1 \implies \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \implies 1 + \tan^2\theta = \sec^2\theta\,.$$

We are given that $\sec\theta = \frac{x}{5}$ and so

$$\tan^2\theta = \sec^2\theta - 1 = \frac{x^2}{5^2} - 1 = \frac{x^2-25}{5^2} = \left(\frac{\pm\sqrt{x^2-25}}{5}\right)^2\,.$$

So $\tan\theta = \pm\frac{\sqrt{x^5-25}}{5}$ and we just now need to determine whether to take the plus or the minus.
Now we can proceed as we did in Way # 1 in (1).

Key Idea: This way shows us how, from a well-known Pythagorean equality $\cos^2\theta + \sin^2\theta = 1$, to derive a Pythagorean equality relating $\tan\theta$ and $\sec\theta$.
Can you derive a Pythagorean equality relating $\cot\theta$ and $\csc\theta$?

Instructions For Remaining $u - du$ Problems 6 – 11:
First show all your work below the box then put answer in the box.
No credit will be given for an answer just put in a box without proper justification.
Box your $u - du$ substitution. Work in a logical fashion.

How to pick u for a $u - du$ substitution? Loosely speaking, we often view an integral as

$$\int f(x)\,dx = \int \text{(a function of } x\text{)}\ [\text{(another function of } x\text{)}\,dx]$$

where the [(another function of x) dx] is essentially (up to a constant) the du. In $u - du$ sub.'s, when picking the u, we do not worry about the constants since a constant just jumps over the integral sign and comes along for the ride. Then we adjust for that constant (by finding another constant K) and write

$$\int f(x)\ dx = K \int \text{(a function of } x\text{)} \boxed{\text{(still another function of } x\text{)}\ dx}$$

where the $\boxed{\text{(still another function of } x\text{)}\ dx}$ is exactly the du.

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6. $\displaystyle \int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = 2\sqrt{1+\sin x}$ $+ C$

View as: $\displaystyle \int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int \frac{1}{\sqrt{1+\sin x}} \boxed{\cos x\,dx}$ with $\boxed{\begin{array}{l} u = 1+\sin x \\ du = \cos x\,dx \end{array}}$.

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{u} + C = 2\sqrt{1+\sin x} + C$$

7. $\displaystyle \int \frac{dx}{x\ln x} = \ln|\ln x|$ $+ C$

View as $\displaystyle \int \frac{dx}{x\ln x} = \int \frac{1}{\ln x} \boxed{\frac{dx}{x}}$ with $\boxed{\begin{array}{l} u = \ln x \\ du = \frac{dx}{x} \end{array}}$ so $\displaystyle \int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$.

8. $\displaystyle \int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}$ $+ C$

Recall order of operations: $e^{-x^2} = e^{\left(-x^2\right)}$.

Way 1 View as: $\displaystyle \int xe^{-x^2}\,dx = \int e^{-x^2}\,[x\,dx] = -\frac{1}{2}\int e^{-x^2}\boxed{-2x\,dx}$ with $\boxed{\begin{array}{l} u = -x^2 \\ du = -2x\,dx \end{array}}$.

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C.$$

Way 2 View as: $\displaystyle \int xe^{-x^2}\,dx = \int 1\left[e^{-x^2}x\,dx\right] = -\frac{1}{2}\int \boxed{e^{-x^2}\,(-2x)\,dx}$ with $\boxed{\begin{array}{l} u = e^{\left(-x^2\right)} \\ du = e^{\left(-x^2\right)}\,(-2x)\,dx \end{array}}$.

$$\int xe^{-x^2}\,dx = -\frac{1}{2}\int du = -\frac{1}{2}u + C = -\frac{1}{2}e^{-x^2} + C.$$

9. $\displaystyle \int \frac{dx}{\sqrt{9-4x^2}} \;=\; \frac{1}{2}\arcsin\left(\frac{2x}{3}\right)$ + C

If you do not recall the formula

$$\int \frac{dx}{\sqrt{a^2-x^2}} \overset{a>0}{=} \sin^{-1}\left(\frac{x}{a}\right) + C$$

but do remember that

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$$

then your awesome algebra can save you since, with $a>0$,

$$\int \frac{dx}{\sqrt{a^2-x^2}} \overset{\text{algebra}}{=} \int \frac{1}{\sqrt{(a^2)\left(1-\frac{x^2}{a^2}\right)}}\,dx \overset{\text{algebra}}{=} \int \frac{1}{\sqrt{a^2}\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx \overset{\text{algebra}}{=} \frac{1}{a}\int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,dx$$

now we let $u=\dfrac{x}{a}$ and so $du=\dfrac{dx}{a}$, which gives

$$= \frac{1}{a}\int \frac{a\,du}{\sqrt{1-u^2}} \overset{\text{algebra}}{=} \frac{a}{a}\int \frac{du}{\sqrt{1-(u)^2}} = \sin^{-1}(u) + C = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

View as: $\displaystyle \int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{3^2-(2x)^2}}\,[dx] = \frac{1}{2}\int \frac{1}{\sqrt{3^2-(2x)^2}}\,\boxed{2\,dx}$ with $\boxed{\begin{array}{l} u=2x \\ du=2\,dx \end{array}}$.

So we have

$$\int \frac{dx}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{3^2-(2x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{3^2-(2x)^2}}\,2\,dx = \frac{1}{2}\int \frac{1}{\sqrt{3^2-u^2}}\,du$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

10. $\displaystyle \int \frac{x\,dx}{\sqrt{9-4x^2}} \;=\; -\frac{1}{4}\sqrt{9-4x^2}$ + C

Compare this integral to the previous integral. In this problem, the x in the numerator is essentially (up to a constant) the derivative of the expression $9+4x^2$ inside the radical.

So we let $\boxed{\begin{array}{l} u=9+4x^2 \\ du=-8x\,dx \end{array}}$ to get

$$\int \frac{x\,dx}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{9-4x^2}}\,[x\,dx] = -\frac{1}{8}\int \frac{1}{\sqrt{9-4x^2}}\,\boxed{-8x\,dx} = -\frac{1}{8}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{8}\int u^{-1/2}du = -\frac{1}{8}\frac{u^{1/2}}{\frac{1}{2}} + C = -\frac{1}{8}\cdot\frac{2}{1}\sqrt{u} + C = -\frac{1}{4}\sqrt{9-4x^2} + C.$$

11. Lastly, an definite integral (i.e., an integral with limits of integration).
The previous integrals were indefinite integral (i.e., an integrals without limits of integration).

$$\int_{x=\pi/6}^{x=\pi/3} \frac{\sin x}{\cos^2 x}\,dx = 2 - \frac{2\sqrt{3}}{3} \overset{\text{also ok}}{=} 2\left(1 - \frac{\sqrt{3}}{3}\right) \overset{\text{also ok}}{=} 2 - \frac{2}{\sqrt{3}} \quad \ldots \text{ and other variations}$$

Consider the indefinite integral (i.e., temporarily ignore the limits of integration).
View as: $\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,[\sin x\,dx] = -\int \frac{1}{\cos^2 x}\boxed{-\sin x\,dx}$ with $\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}}$.
If $x = \frac{\pi}{6}$, then $u = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.
If $x = \frac{\pi}{3}$, then $u = \cos\frac{\pi}{3} = \frac{1}{2}$.

Way 1 First let's do the indefinite integral $\int \frac{\sin x}{\cos^2 x}\,dx$.

$$\int \frac{\sin x}{\cos^2 x}\,dx = -\int \frac{1}{u^2}\,du = -\int u^{-2}\,du = -\frac{u^{-1}}{-1} + C = u^{-1} + C = (\cos x)^{-1} \overset{\text{or}}{=} \sec x + C.$$

Then check your indefinite integral by differentiating your answer to the indefinite integral to be sure you get the integrand $\frac{\sin x}{\cos^2 x}$.
This key concept is the Fundemental Theorem of Calculus (FTC) – in action!

$$D_x(\cos x)^{-1} = -(\cos x)^{-2}\,(D_x \cos x) = -(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} \qquad \boxed{\checkmark}.$$

Then plug in your limits of integration.

$$\int_{x=\pi/6}^{x=\pi/3} \frac{\sin x}{\cos^2 x}\,dx = (\cos x)^{-1}\Big|_{x=\pi/6}^{x=\pi/3} = \frac{1}{\cos\frac{\pi}{3}} - \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} - \frac{1}{\frac{\sqrt{3}}{2}} = 2 - \frac{2}{\sqrt{3}}$$

$$= 2 - \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = 2 - \frac{2\sqrt{3}}{3}.$$

Way 2

$$\int_{x=\pi/6}^{x=\pi/3} \frac{\sin x}{\cos^2 x}\,dx = -\int_{x=\pi/6}^{x=\pi/3} \frac{1}{\cos^2 x}\boxed{-\sin x\,dx} = -\int_{u=\frac{\sqrt{3}}{2}}^{u=1/2} \frac{1}{u^2}\,du = -\int_{u=\frac{\sqrt{3}}{2}}^{u=1/2} u^{-2}\,du$$

$$= -\frac{u^{-1}}{-1}\Big|_{u=\frac{\sqrt{3}}{2}}^{u=1/2} = \frac{1}{u}\Big|_{u=\frac{\sqrt{3}}{2}}^{u=1/2} = \frac{1}{\frac{1}{2}} - \frac{1}{\frac{\sqrt{3}}{2}} = 2 - \frac{2}{\sqrt{3}}.$$

Wrong Way

$$\int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx = -\int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x}\boxed{-\sin x\,dx} = -\int_{\pi/6}^{\pi/3} \frac{1}{u^2}\,du = -\int_{\pi/6}^{\pi/3} u^{-2}\,du$$

$$= -\frac{u^{-1}}{-1}\Big|_{\pi/6}^{\pi/3} = \frac{1}{u}\Big|_{\pi/6}^{\pi/3} = \frac{1}{\frac{\pi}{3}} - \frac{1}{\frac{\pi}{6}} = \frac{3}{\pi} - \frac{6}{\pi} = \frac{-3}{\pi}.$$

The mistake is $-\int_{\pi/6}^{\pi/3} \frac{1}{\cos^2 x}\boxed{-\sin x\,dx} \neq -\int_{\pi/6}^{\pi/3} \frac{1}{u^2}\,du$ since when one changes variables (from x to u) in the integrand, one also needs to change the limits of integration (from x to u).

Question. Why does using **Way 1** tend to lead to fewer mistakes?