1. Determine (all) values of $p \in \mathbb{R}$ for which the following series converges. HINT: Integral Test.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

ANSWER: The series $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ converges if and only if $\underline{\qquad\qquad p > 1 \qquad\qquad}$.
Fill in the line with some condition on p and then *justify your answer below.*

If $p \leq 0$, then $\lim_{n\to\infty} \frac{\ln n}{n^p} = \infty$ and so, by the n^{th} term test, $\sum \frac{\ln n}{n^p}$ diverges. Thus

$$\boxed{\sum_{n=1}^{\infty} \frac{\ln n}{n^p} \quad \text{diverges when} \quad p \leq 0}\ .$$

Now let $p > 0$. Let's check the conditions of the Integral Test. So define $f\colon [1,\infty) \to \mathbb{R}$ by $f(x) = \frac{\ln x}{x^p}$.

(1) We were given $a_n = \frac{\ln n}{n^p}$ and so $f(n) = a_n$ for all $n \in \mathbb{N}$ (this was by design)

(2) $y = f(x)$ positive on $[1,\infty)$? yes - clear

(3) $y = f(x)$ continuous on $[1,\infty)$? yes - clear

(4) $y = f(x)$ decreasing on $[1,\infty)$? maybe - let's use the 1st-derivative test to check if it is decreasing.

Checking if f is decreasing on $[1,\infty)$: for $x \in [1,\infty)$

$$f'(x) = \frac{\frac{1}{x}(x^p) - (\ln x)\, p x^{p-1}}{(x^p)^2} \overset{Ⓐ}{=} \frac{x^{p-1}(1 - p\ln x)}{x^{2p}} < 0 \iff 1 - p\ln x < 0 \overset{Ⓐ}{\iff} e^{\frac{1}{p}} < x\ .$$

So f is decreasing on $[e^{1/p},\infty)$. So f satisfies the conditions of the integral test on the interval $[e^{1/p},\infty)$. So by the integral test, the improper integral $\int_{e^{1/p}}^{\infty} \frac{\ln x}{x^p}\, dx$ and the infinite series $\sum \frac{\ln n}{n^p}$ *do the same thing* (i.e., either both converge or both diverge). So let's investiage this improper integral.
By basic calculus (which you should show)

$$\int \frac{\ln x}{x^p}\, dx = \begin{cases} \frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2} & p \neq 1 \text{ (use integration by parts with } u = \ln x) \\ \frac{(\ln x)^2}{2} & p = 1 \text{ (use substitution with } u = \ln x)\ . \end{cases} \tag{1}$$

So if $p = 1$, then

$$\int_{e^{1/p}}^{b} \frac{\ln x}{x^p}\, dx = \int_{e^{1/p}}^{b} \frac{\ln x}{x}\, dx \overset{\text{by (1)}}{=} \left.\frac{(\ln x)^2}{2}\right|_{x=e^{1/p}}^{x=b} = \frac{(\ln b)^2}{2} - \frac{(\ln e^{1/p})^2}{2} = \frac{(\ln b)^2}{2} - \frac{1}{2p^2} \xrightarrow{b\to\infty} \infty.$$

If $p \neq 1$ (recall we still have $p > 0$), then

$$\int_{e^{1/p}}^{t} \frac{\ln x}{x^p}\, dx \overset{\text{by (1)}}{=} \left.\frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2}\right|_{x=e^{1/p}}^{x=t} = \frac{t^{1-p}[(1-p)\ln t - 1]}{(1-p)^2} - \underbrace{\frac{\left(e^{1/p}\right)^{1-p}[(1-p)\ln e^{1/p} - 1]}{(1-p)^2}}_{\text{some finite constant, all we care about is that this constant is finite}}$$

So we need to evaluate the behavior of $\lim_{t\to\infty} t^{1-p}[(1-p)\ln t - 1]$, which depends on whether $1-p$ (i.e., the power to which t is raised) is positive or negative.

Case $p > 1$. So $1 - p < 0$ and $0 < p - 1$.

$$\lim_{t\to\infty} t^{1-p}[(1-p)\ln t - 1] \overset{Ⓐ}{=} \lim_{t\to\infty} \frac{(1-p)\ln t - 1}{t^{p-1}} \underset{\text{L'H}}{\overset{\frac{\infty}{\infty}}{=}} \lim_{t\to\infty} \frac{(1-p)\,t^{-1}}{(p-1)\,t^{p-1-1}} \overset{Ⓐ}{=} \lim_{t\to\infty} \frac{-1}{t^{p-1}} \overset{p-1>0}{=} 0\ .$$

Case $0 < p < 1$. So $0 < 1 - p < 1$. Then

$$\lim_{t\to\infty} t^{1-p} \overset{1-p>0}{=} \infty \quad \text{and} \quad \lim_{t\to\infty} \ln t = \infty$$

and so

$$\lim_{t\to\infty} t^{1-p}\left[(1-p)\ln t - 1\right] = \infty \ .$$

Conclusion for $p > 0$. The improper integral

$$\int_{e^{1/p}}^{\infty} \frac{\ln x}{x^p}\,dx \quad \text{is} \quad \begin{cases} \text{divergent} & \text{if } 0 < p \le 1 \\ \text{convergent} & \text{if } p > 1 \ . \end{cases}$$

Thus we have shown

$$\boxed{\sum_{n=1}^{\infty} \frac{\ln n}{n^p}\,dx \quad \text{is} \quad \begin{cases} \text{divergent} & \text{if } 0 < p \le 1 \\ \text{convergent} & \text{if } p > 1 \ . \end{cases}}$$

2. Find the sum of the (telescoping) series

$$\sum_{n=1}^{\infty} \frac{2}{n^2+2n} \ .$$

HINT: $\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}$.

Let $a_n = \frac{2}{n^2+2n}$ and $s_n = \sum_{k=1}^{n} a_k$. Thus we want to consider $\lim_{n\to\infty} s_n$. The hint (thanks) gives us the the Partial Fractions Decomposition of a_n:

$$\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2} \ .$$

Since the denominators n and $n+2$ of the PFD differ by 2, we will write 2 terms per line

$$\begin{aligned}
s_n = {} & \frac{+1}{1} + \frac{-\cancel{1}}{\cancel{3}} + \frac{+1}{2} + \frac{-\cancel{1}}{\cancel{4}} & \rightsquigarrow k = 1 \text{ and } k = 2 \text{ terms} \\
& \quad ① \swarrow \qquad ② \swarrow & \\
+ {} & \frac{+\cancel{1}}{\cancel{3}} + \frac{-\cancel{1}}{\cancel{5}} + \frac{+\cancel{1}}{\cancel{4}} + \frac{-\cancel{1}}{\cancel{6}} & \rightsquigarrow k = 3 \text{ and } k = 4 \text{ terms} \\
& \quad ③ \swarrow \qquad ④ \swarrow & \\
+ {} & \frac{+\cancel{1}}{\cancel{5}} + \frac{-\cancel{1}}{\cancel{7}} + \frac{+\cancel{1}}{\cancel{6}} + \frac{-\cancel{1}}{\cancel{8}} & \rightsquigarrow k = 5 \text{ and } k = 6 \text{ terms} \\
& \quad ⑤ \swarrow \qquad ⑥ \swarrow & \\
+ {} & \frac{+\cancel{1}}{\cancel{7}} + \frac{-\cancel{1}}{\cancel{9}} + \frac{+\cancel{1}}{\cancel{8}} + \frac{-\cancel{1}}{\cancel{10}} & \rightsquigarrow k = 7 \text{ and } k = 8 \text{ terms} \\
\vdots \; & \quad ⑦ \swarrow \qquad ⑧ \swarrow & \\
+ {} & \frac{+\cancel{1}}{\cancel{n-3}} + \frac{-\cancel{1}}{\cancel{n-1}} + \frac{+\cancel{1}}{\cancel{n-2}} + \frac{-\cancel{1}}{\cancel{n}} & \rightsquigarrow k = n-3 \text{ and } k = n-2 \text{ terms} \\
& \quad \swarrow \qquad \swarrow & \\
+ {} & \frac{+\cancel{1}}{\cancel{n-1}} + \frac{-1}{n+1} + \frac{+\cancel{1}}{\cancel{n}} + \frac{-1}{n+2} & \rightsquigarrow k = n-1 \text{ and } k = n \text{ terms} \ .
\end{aligned}$$

Thus we have

$$s_n = 1 + \frac{1}{2} + \frac{-1}{n+1} + \frac{-1}{n+2} \xrightarrow{n\to\infty} 1 + \frac{1}{2} + 0 + 0 = \frac{3}{2} \ .$$

Thus the answer is: The series $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ converges to the sum $\boxed{\frac{3}{2}}$.