Math 141 HANDOUT

Fundamental Theorem of Calculus (FTC)

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function.
Let $F\colon [a,b] \to \mathbb{R}$ be a function.

- If F is an antiderivative of f on $[a,b]$ (i.e. $F'(x) = f(x)$ for each $x \in [a,b]$), then

$$\int_a^b f(x)\,dx \;\equiv\; \int_a^b F'(x)\,dx \;=\; F(b) - F(a)\ .$$

- If $F(x) = \int_a^x f(t)\,dt$ for each $x \in [a,b]$, then F is an antiderivative of f on $[a,b]$, i.e.

$$F'(x) \equiv D_x\left[\int_a^x f(t)\,dt\right] \;=\; f(x)\ .$$

Basic Differentiation Rules

If the functions $y = f(x)$ and $y = g(x)$ are differentiable at x and a and b are constants, then:

1. $D_x\left[af(x) + bg(x)\right] = af'(x) + bg'(x)$
2. $D_x\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
3. $D_x\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{[g(x)]^2}$ provided $g(x) \neq 0$
4. $D_x\left[f(x)\right]^r = r\left[f(x)\right]^{r-1} f'(x)$ provided $r \in \mathbb{Q}$

If f is differentiable at x and g is differentiable at $f(x)$, then:

5. $D_x\left[g(f(x))\right] = g'(f(x))\, f'(x)$

Generalized Exponential and Logarithmic Functions with base a where $a > 0$ but $a \neq 1$

DERIVATIVES	$\overset{\text{FTC}}{\Longrightarrow}$	INTEGRALS
$D_x \ln\lvert u\rvert \overset{u\neq 0}{=} \frac{1}{u}\frac{du}{dx}$		$\int \frac{du}{u} \overset{u\neq 0}{=} \ln\lvert u\rvert + C$
$D_x\, e^u = e^u \frac{du}{dx}$		$\int e^u\,du = e^u + C$
$D_x \log_a\lvert u\rvert \overset{u\neq 0}{=} \frac{1}{u}\,\frac{1}{\ln a}\,\frac{du}{dx}$		
$D_x a^u = a^u \ln a \frac{du}{dx}$		$\int a^u\,du = \frac{a^u}{\ln a} + C$

TRIG and CALCULUS

DERIVATIVES $\overset{\text{FTC}}{\Longrightarrow}$ INTEGRALS

$$D_x \sin u = \cos u \, \frac{du}{dx} \qquad \int \cos u \, du = \sin u + C$$

$$D_x \tan u = \sec^2 u \, \frac{du}{dx} \qquad \int \sec^2 u \, du = \tan u + C$$

$$D_x \sec u = \sec u \tan u \, \frac{du}{dx} \qquad \int \sec u \tan u \, du = \sec u + C$$

$$D_x \cos u = -\sin u \, \frac{du}{dx} \qquad \int \sin u \, du = -\cos u + C$$

$$D_x \cot u = -\csc^2 u \, \frac{du}{dx} \qquad \int \csc^2 u \, du = -\cot u + C$$

$$D_x \csc u = -\csc u \cot u \, \frac{du}{dx} \qquad \int \csc u \cot u \, du = -\csc u + C$$

MORE INTEGRALS

$$\int \tan u \, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C = -\ln|\csc u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C = -\ln|\sec u - \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C = \ln|\csc u - \cot u| + C$$

DERIVATIVES $\overset{\text{FTC}}{\Longrightarrow}$ INTEGRALS $(a > 0)$

$$D_x \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx} \qquad \int \frac{1}{\sqrt{a^2-u^2}} \, du = \sin^{-1} \frac{u}{a} + C$$

$$D_x \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx} \qquad \int \frac{1}{a^2+u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$D_x \sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dx} \qquad \int \frac{1}{u\sqrt{u^2-a^2}} \, du = \frac{1}{a} \sec^{-1} \frac{|u|}{a} + C$$

$$D_x \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$D_x \cot^{-1} u = \frac{-1}{1+u^2} \frac{du}{dx}$$

$$D_x \csc^{-1} u = \frac{-1}{|u|\sqrt{u^2-1}} \frac{du}{dx}$$

Natural Logarithm Fn. $y = \ln x$ AND Natural Exponential Fn. $y = \exp x$

$\ln x \overset{x>0}{\equiv} \int_1^x \frac{dt}{t}$

$\ln\colon (0,\infty) \to (-\infty,\infty)$

$y = \ln x \iff x = \exp y$

$\exists$ a unique $e \in \mathbb{R}$ so that $\ln e = 1$

nat. exp. fn. $\equiv$ inverse of the nat. log. fn.

$\exp\colon (-\infty,\infty) \to (0,\infty)$

$e^x \overset{x\in\mathbb{R}}{\equiv} \exp(x)$

$x, y > 0$ & $r \in \mathbb{Q}$	$x, y \in \mathbb{R}$ & $r \in \mathbb{Q}$
$e^{\ln x} = x$	$\ln(e^x) = x$
$\ln 1 = 0$	$e^0 = 1$
$\ln(xy) = \ln x + \ln y$	$e^x\, e^y = e^{x+y}$
$\ln(\frac{x}{y}) = \ln x - \ln y$	$\frac{e^x}{e^y} = e^{x-y}$
$\ln(x^r) = r(\ln x)$	$(e^x)^r = e^{xr}$

Generalized Exponential $y = a^x$ and Logarithmic $y = \log_a x$ Functions

with base a where $a > 0$ but $a \neq 1$ and $b > 0$ but $b \neq 1$

$\log_e \equiv \ln$

$f(x) = a^x \equiv e^{x \ln a} \quad : (-\infty,\infty) \to (0,\infty)$

$g(x) = \log_a x \equiv$ the inverse of the fn. $f(x) = a^x \quad : (0,\infty) \to (-\infty,\infty)$

$y = \log_a x \iff x = a^y$

$(\log_a b)(\log_b c) = \log_a c \implies \log_a x = \frac{\ln x}{\ln a}$

$x, y > 0$ & $r \in \mathbb{R}$	$x, y \in \mathbb{R}$ & $r \in \mathbb{R}$ & $0 < b \neq 1$
$a^{\log_a x} = x$	$\log_a(a^x) = x$
$\log_a 1 = 0$	$a^0 = 1$
$\log_a(xy) = \log_a x + \log_a y$	$a^x\, a^y = a^{x+y}$
$\log_a(\frac{x}{y}) = \log_a x - \log_a y$	$\frac{a^x}{a^y} = a^{x-y}$
$\log_a(x^r) = r(\log_a x)$	$(a^x)^r = a^{xr}$
	$(ab)^x = a^x b^x$ and $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Basic Trig

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Basic Inverse Trig Functions

$y = \sin\theta$	$\Leftrightarrow$	$\theta = \sin^{-1} y$	where	$-1 \le y \le 1$	and	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
$y = \cos\theta$	$\Leftrightarrow$	$\theta = \cos^{-1} y$	where	$-1 \le y \le 1$	and	$0 \le \theta \le \pi$
$y = \tan\theta$	$\Leftrightarrow$	$\theta = \tan^{-1} y$	where	$y \in \mathbb{R}$	and	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$y = \cot\theta$	$\Leftrightarrow$	$\theta = \cot^{-1} y$	where	$y \in \mathbb{R}$	and	$0 < \theta < \pi$
$y = \sec\theta$	$\Leftrightarrow$	$\theta = \sec^{-1} y$	where	$\lvert y\rvert \ge 1$	and	$0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$
$y = \csc\theta$	$\Leftrightarrow$	$\theta = \csc^{-1} y$	where	$\lvert y\rvert \ge 1$	and	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$

Math 142

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Trig Identities: (∗) do not have to memorize but should be able to use if given

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \sin(2x) = 2\sin x\cos x$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos(s+t) \overset{(*)}{=} \cos s\cos t - \sin s\sin t \qquad \sin(s+t) \overset{(*)}{=} \sin s\cos t + \cos s\sin t$$

$$\cos(s-t) \overset{(*)}{=} \cos s\cos t + \sin s\sin t \qquad \sin(s-t) \overset{(*)}{=} \sin s\cos t - \cos s\sin t$$

Trig Substitution

IF INTEGRAND INVOLVES	THEN MAKE THE SUBSTITUTION	RESTRICTION ON θ
$a^2 - u^2$	$u = a\sin\theta \iff \theta = \sin^{-1}\frac{u}{a}$	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
$a^2 + u^2$	$u = a\tan\theta \iff \theta = \tan^{-1}\frac{u}{a}$	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$u^2 - a^2$	$u = a\sec\theta \iff \theta = \sec^{-1}\frac{u}{a}$	$0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$

INFINITE SERIES – SUMMARY

GEOMETRIC SERIES: WITH RATIO r (AND $c \neq 0$)

$$\sum_{n=0}^{\infty} c\,r^n = c(1+r+r^2+r^3+r^4+\ldots) = \begin{cases} \text{converges} & |r|<1 \\ \text{diverges} & |r|\geqslant 1 \end{cases}$$

Since $\sum_{n=0}^{N} r^n \equiv s_N = \frac{1-r^{N+1}}{1-r}$.

p-SERIES

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^p = \sum_{n=1}^{\infty}\frac{1}{n^p} = 1+\left(\frac{1}{2}\right)^p+\left(\frac{1}{3}\right)^p+\left(\frac{1}{4}\right)^p+\ldots = \begin{cases} \text{converges} & p>1 \\ \text{diverges} & p\leqslant 1 \end{cases}$$

Show this via Integral Test.
If $p=1$, it's called the harmonic series.

n^{th}-TERM TEST FOR DIVERGENCE

The Test: If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ DNE, then $\sum a_n$ diverges.
Because: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
Warning: If $\lim_{n\to\infty} a_n = 0$, then it is possible that $\sum a_n$ converges and it is possible that $\sum a_n$ diverges.
Remark: The n^{th} can show divergence but can NOT show convergence.

DEFINITIONS

$\sum a_n$ is absolutely convergent $\iff$ [$\sum |a_n|$ converges]
$\sum a_n$ is conditionally convergent $\iff$ [$\sum |a_n|$ diverges AND $\sum a_n$ converges]
$\sum a_n$ is divergent $\iff$ [$\sum a_n$ diverges]

BIG THEOREM

Theorem:
If $\sum |a_n|$ converges, then $\sum a_n$ converges.
So we get for free:
If $\sum a_n$ diverges, then $\sum |a_n|$ diverges.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE POSSIBILITIES

$\sum a_n$ is absolutely convergent $\iff$ [$\sum |a_n|$ converges $\overset{\text{implies}}{\Longrightarrow}$ $\sum a_n$ converges]
$\sum a_n$ is conditionally convergent $\iff$ [$\sum |a_n|$ diverges AND $\sum a_n$ converges]
$\sum a_n$ is divergent $\iff$ [$\sum a_n$ diverges $\overset{\text{implies}}{\Longrightarrow}$ $\sum |a_n|$ diverges]

PROBLEM: we need to figure out if an infinite series $\sum a_n$ is: absolutely convergent, conditionally convergent, or divergent.
SOLUTION: we apply one of the below TESTS that will give us the answer. Which one … well, pattern recognition time. Sometimes more than one test will work! For some of the tests, we need to find the appropriate $\sum b_n$, which is usually a well-known series (like a geometric series or p-series) that we know whether it converges or diverges.

NAME	STATEMENT OF TEST
	POSITIVE-TERMED SERIES TESTS $\sum a_n$ where $a_n \geqslant 0$ $\forall n \in \mathbb{N}$
Key Idea	$\sum a_n$ converge $\iff \{s_N\}_{N=1}^{\infty}$ is bounded above (since $a_n \geqslant 0 \iff s_n \nearrow$)
Integral Test	Let $f\colon [1,\infty) \to \mathbb{R}$ be continuous, positive, and nonincreasing function with $f(n) = a_n \forall n \in \mathbb{N}$. Then [$\sum a_n$ converges $\iff \int_1^{\infty} f(x)\,dx$ converges] .
Comparison Test (CT)	[$0 \leqslant a_n \leqslant b_n$ $\forall n \geqslant N_0$ & $\sum b_n$ conv.] $\Longrightarrow$ [$\sum a_n$ conv.] [$0 \leqslant b_n \leqslant a_n$ $\forall n \geqslant N_0$ & $\sum b_n$ divg.] $\Longrightarrow$ [$\sum a_n$ divg.]
Limit Comparison Test (LCT)	Let $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. If $0 < L < \infty$, then [$\sum a_n$ conv. $\iff \sum b_n$ conv.] (you DO need to memorize this one) If $L = 0$, then [$\sum b_n$ conv. $\Longrightarrow \sum a_n$ conv.] (you do not have to memorize this one) If $L = \infty$, then [$\sum b_n$ divg. $\Longrightarrow \sum a_n$ divg.] (you do not have to memorize this one)
	ARBITRARY-TERMED SERIES TESTS $\sum a_n$ where $-\infty < a_n < \infty$ $\forall n \in \mathbb{N}$
Ratio Test	Let $\rho = \lim_{n\to\infty} \left\lvert\frac{a_{n+1}}{a_n}\right\rvert$.
Root Test	Let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} \overset{\text{note}}{=} \lim_{n\to\infty} \lvert a_n\rvert^{\frac{1}{n}}$. $0 \leqslant \rho < 1 \Longrightarrow \sum a_n$ converges absolutely $\rho = 1 \Longrightarrow$ test is inconclusive $1 < \rho \leqslant \infty \Longrightarrow \sum a_n$ diverges (by n^{th} term test for divergence)
	ALTERNATING SERIES TEST $\sum a_n = \sum(-1)^n u_n$ where $u_n > 0$ $\forall n \in \mathbb{N}$, in other words $a_n = (-1)^n u_n$ and $u_n > 0$
Alternating Series Test (AST)	[$u_n > u_{n+1}$ $\forall n \in \mathbb{N}$ & $\lim_{n\to\infty} u_n = 0$] $\Longrightarrow$ [$\sum a_n = \sum(-1)^n u_n$ conv.]

0. Fill in: the boxes in problem **0A** and the lines in problem **0B**.

0A. Taylor/Maclaurin Polynomials and Series

Let $y = f(x)$ be a function with derivatives of all orders in an interval I containing x_0.

Let $y = P_N(x)$ be the N^{th}-order Taylor polynomial of $y = f(x)$ about x_0.

Let $y = R_N(x)$ be the N^{th}-order Taylor remainder of $y = f(x)$ about x_0.

Let $y = P_\infty(x)$ be the Taylor series of $y = f(x)$ about x_0.

Let c_n be the n^{th} Taylor coefficient of $y = f(x)$ about x_0.

In open form (i.e., with ... and without a $\sum$-sign)

$$P_N(x) \;=\;= f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N$$

In closed form (i.e., with a $\sum$-sign and without ...)

$$P_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

In open form (i.e., with ... and without a $\sum$-sign)

$$P_\infty(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \ldots$$

In closed form (i.e., with a $\sum$-sign and without ...)

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

We know that $f(x) = P_N(x) + R_N(x)$. Taylor's BIG Theorem tells us that, for each $x \in I$,

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{(N+1)}$$

for some c between x and x_0.

The formula for c_n is

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

A Maclaurin series is a Taylor series about the center $x_0 = 0$.

0B. Volume of Revolutions. Let's say we revolve some region in the xy-plane around an axis of revolution so we get a solid of revolution. Next we want to find the volume of this solid of revolution.

- **In parts a, fill in the blanks with:** x **or** y.
- **In parts b and c, fill in the blanks with a formula involving *some of*:** **2 , π , radius , radius$_{\text{big}}$, radius$_{\text{little}}$, average radius , height , and/or thickness .**

▶. **Disk/Washer Method**. Let's find the volume of this solid of revolution using the disk or washer method.

a. If the axis of revolution is:

- the x-axis, or parallel to the x-axis, then we partition the $\underline{\ x\ }$–axis.
- the y-axis, or parallel to the y-axis, then we partition the $\underline{\ y\ }$–axis.

b. If we use the **disk method**, then the volume of a typical disk is:

π (radius)^2 (height) .

If we use the **washer method**, then the volume of a typical washer is:

π (radius$_{\text{big}}$)^2 (height) $-$ π (radius$_{\text{little}}$)^2 (height) $\stackrel{\text{or}}{=}$ π $\left[(\text{radius}_{\text{big}})^2 - (\text{radius}_{\text{little}})^2\right]$ (height) .

c. If we partition the z-axis, where z is either x or y, the $\Delta z =$ height .

▶. **Shell Method**. Let's find the volume of this solid of revolution using the shell method.

a. If the axis of revolution is:

- the x-axis, or parallel to the x-axis, then we partition the $\underline{\ y\ }$–axis.
- the y-axis, or parallel to the y-axis, then we partition the $\underline{\ x\ }$–axis.

b. If we use the **shell method**, then the volume of a typical shell is:

2π (average radius) (height) (thickness) $\stackrel{\text{or}}{=}$ 2π (radius) (height) (thickness) .

c. If we partition the z-axis, where z is either x or y, the $\Delta z =$ thickness $\stackrel{\text{or}}{=}$ radius$_{\text{big}}$ $-$ radius$_{\text{little}}$.

Commonly Used Taylor Series

		SERIES	WHEN IS VALID/TRUE
$\frac{1}{1-x}$	=	$1 + x + x^2 + x^3 + x^4 + \ldots$	NOTE THIS IS THE GEOMETRIC SERIES. JUST THINK OF x AS r
	=	$\sum_{n=0}^{\infty} x^n$	$x \in (-1, 1)$
e^x	=	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$	SO: $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$ $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$
	=	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$x \in \mathbb{R}$
$\cos x$	=	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$	NOTE $y = \cos x$ IS AN <u>EVEN</u> FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY <u>EVEN</u> POWERS.
	=	$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$x \in \mathbb{R}$
$\sin x$	=	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$	NOTE $y = \sin x$ IS AN <u>ODD</u> FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY <u>ODD</u> POWERS.
	=	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x \in \mathbb{R}$
$\ln(1+x)$	=	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$	QUESTION: IS $y = \ln(1+x)$ EVEN, ODD, OR NEITHER?
	=	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^n}{n} \overset{\text{or}}{=} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$	$x \in (-1, 1]$
$\tan^{-1} x$	=	$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$	QUESTION: IS $y = \arctan(x)$ EVEN, ODD, OR NEITHER?
	=	$\sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{2n-1} \overset{\text{or}}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$x \in [-1, 1]$

Polar Coordinate

Conversion

A point P in the plane with:
Cartesian coordinates (x, y) and polar coordinates (r, θ) satisfies the following.

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}\,.$$

Graphing

The period of $y = \sin(k\theta)$ and of $y = \cos(k\theta)$ is $\frac{2\pi}{k}$.
To sketch graphs, divide the period by 4.

Area

Let $A(r, \theta)$ be the area of a sector of a circle with cental angle θ radians and radius r.
Comparing $A(r, \theta)$ to the area of the whole circle lead us to the proportion:

$$\frac{A(r,\theta)}{A(r,2\pi)} = \frac{\theta}{2\pi} \quad\Rightarrow\quad \frac{A(r,\theta)}{\pi r^2} = \frac{\theta}{2\pi} \quad\Rightarrow\quad A(r,\theta) = \frac{\theta}{2\pi}\,\frac{\pi r^2}{1}$$

from which we concluded that

$$A(r,\theta) = \frac{\theta r^2}{2}\,.$$

Now consider a function $r = f(\theta)$ which determines a curve in the plane where

(1) $f\colon [\alpha, \beta] \to [0, \infty]$
(2) f is continuous on $[\alpha, \beta]$
(3) $\beta - \alpha \leq 2\pi$.

Then the area bounded by $r = f(\theta)$ and $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} [f(\theta)]^2\, d\theta\,.$$