PROJECT # 3 COMPLEX NUMBERS & GEOMETRY

(1) assigned: Wednesday 10 November 1993
(2) due: Tuesday 23 November 1993
(3) At the end of class, form a group of 3 students and report your group members to me. You may NOT work with anyone from your Project # 1 group nor your Project # 2 group.
(4) Since this is the third project, you do not have to present a rough draft.
(5) Prof. Girardi, Mr. Ho, and the Math Lab will answer your project questions through Wednesday 17 November. After that, we are no longer available for help. You are then on your own.
(6) Remember, your project presentation should read even better than our textbook. Your answers should be in complete sentences. Explain your thoughts! Enjoy!

We all know the famous 3-4-5 right triangle. Are there other right triangles with sides of integer lengths? We will call such a triangle an *integral right triangle*, or just IRT for short. The purpose of this project is to show that there are lots of IRT.

Note that

$$3^2 + 4^2 = 5^2 \tag{A}$$

and if n is any constant, then

$$(3n)^2 + (4n)^2 = (5n)^2 . \tag{B}$$

So if n is a positive integer, then the right triangle with sides of length $3n-4n-5n$ is another IRT. So we think of our favorite $3-4-5$ IRT as producing a whole *family of IRT*, namely all those IRT with sides of length $3n-4n-5n$ for some positive integer n. If it helps, you can think of the $3-4-5$ *family of IRT* as a sequence of triples:

$$\{3-4-5,\ 6-8-10,\ 9-12-15,\ 12-16-20,\ 15-20-25,\ 18-24-30,\ \ldots\} ,$$

with each triple representing the lenghts of the sides of an IRT. Dividing equation (A) through by 5^2, or likewise, dividing equation (B) through by $(5n)^2$ gives the same equation:

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1 . \tag{C}$$

So when we look for a *family of IRT*, we really can just look for rational numbers x and y such that

$$x^2 + y^2 = 1 .$$

For example, the pair $\left(\frac{3}{5}, \frac{4}{5}\right)$ of rational numbers gives rise to the $3-4-5$ IRT and also the $3-4-5$ *family of IRT*. We will now develop **another** clever method to use the $3-4-5$ IRT to produce many more IRT.

(1) In section 18.3, read from the top of page 903 through the middle of page 907. Do exercises 4,8 & 10 from page 908. Present your solutions

as you would like your book to present it in an examples. Explain your steps. Draw a diagram for 8&10.

(2) Consider the two complex numbers

$$z_1 = x_1 + iy_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = x_2 + iy_2 = r_2 e^{i\theta_2} .$$

Let $z_3 = z_1 \cdot z_2$ First write z_3 in standard form, i. e. find x_3 & y_3 such that $z_3 = x_3 + iy_3$. Next write z_3 in "fancy" polar form, i. e. find $r_3 \geq 0$ and θ_3 such that $z_3 = r_3 e^{i\theta_3}$. Display formulas (7) & (16) from section 18.3 might help.

(3) Now let x_1, x_2, y_1 & y_2 be rational numbers and let $r_1 = r_2 = 1$. Explain why x_3 & y_3 are also rational numbers. What is r_3? Draw the unit circle in the complex plane (see figures 18.4 & 18.5 from the text) and illustrate this phenomenon (esp. the relationship between the θ_i's) of multiplication.

(4) Now consider

$$z_1 = \frac{3}{5} + i\frac{4}{5} .$$

Express z_1 in fancy polar form, i. e. find $r_1 \geq 0$ and θ_1 such that

$$z_1 = r_1 e^{i\theta_1} .$$

First find the *exact* value of θ_1 (i. e. express θ_1 as the inverse cosine of some number) and then find an *approximation* to θ_1 (i. e. ask your calculator for a decimal approximation of this inverse cosine).

(5) Let $z_2 = (z_1)^2$ and $z_3 = (z_1)^3$. Express both z_2 and z_3 in both standard form and fancy polar form. Draw the unit circle in the complex plane and plot z_1, z_2 & z_3 and illustrate this phenomenon of multiplication.

(6) To what IRT does z_2 give rise? Is this IRT in the same *family of IRT* as the 3-4-5 IRT? To what IRT does z_3 give rise? Is this IRT in the same *family of IRT* as the 3-4-5 IRT or the above new IRT?

(7) Using the observation that

$$5^2 \; + \; 12^2 \; = \; 13^2$$

find 2 more *families of IRT*. Be sure to verify that these are truly **different** families of IRT then any other one thus far.

(8) Another neat use of complex numbers:
Using the display formula (16) from the book which gives that

$$e^{i\theta} = \cos\theta \; + \; i\sin\theta ,$$

and the fact that two complex numbers

$$z_1 = x_1 + iy_1 \qquad \text{and} \qquad z_2 = x_2 + iy_2$$

are equal if and only if $x_1 = x_2$ and $y_1 = y_2$, derive the following two trigonometic identities:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta .$$