THE IRREDUCIBILITY OF THE BESSEL POLYNOMIALS

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Dedicated to the memory of Emil Grosswald

1. Introduction

In 1951, Emil Grosswald [7] began investigating the irreducibility of the Bessel Polynomials

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{2^j (n-j)! j!} x^j.$$

He conjectured (cf. [8],[9]) that $y_n(x)$ is irreducible over the rationals for all positive integers n. In this paper, we resolve this conjecture and establish the following generalization.

Theorem. *Let n be a positive integer, and let $a_0, a_1, \dots, a_n$ be arbitrary integers with $|a_0| = |a_n| = 1$. Then*

$$\sum_{j=0}^{n} a_j \frac{(n+j)!}{2^j (n-j)! j!} x^j$$

is irreducible over the rationals.

The above theorem was established in the case that n is sufficiently large, say $n \geq n_0$, by the first author in [4]. He also conjectured there that the above general theorem holds. Although the method in [4] gives an effectively computable value for n_0, a direct application of the methods there does not allow one to establish even that $n_0 \leq 10^{10^{1000}}$. Nevertheless, our approach in this paper is quite similar to that given in [4] and is based on refining the estimates made there. Related work has been done by I. Schur [11], the first author [2,3,5], and S. Graham and the first author [6].

2. Preliminaries

As in previous approaches, we define

$$z_n(x) = x^n y_n(2/x) = \sum_{j=0}^{n} \frac{(2n-j)!}{j!(n-j)!} x^j = \sum_{j=0}^{n} \frac{(n+j)!}{j!(n-j)!} x^{n-j}.$$

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The polynomial $F(x)$ given in the theorem is easily seen to be irreducible if and only if

$$f(x) = x^n F(2/x) = \sum_{j=0}^{n} a_{n-j} \frac{(2n-j)!}{j!(n-j)!} x^j \tag{1}$$

is irreducible. We therefore concentrate our efforts on showing $f(x)$ is irreducible (though with some modifications one can work directly with $F(x)$).

Given $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p a prime, we define the Newton polygon of $g(x)$ with respect to p as the lower convex hull of the points $(j, \nu(b_{n-j}))$ where $\nu(m) = \nu_p(m)$ is the nonnegative integer r for which $p^r \| m$ (cf. [4], [8]; the points where $b_{n-j} = 0$ need not be considered). Our first lemma below is used to connect the degrees of possible factors of the general polynomial $f(x)$ with information about the Newton polygons of $z_n(x)$. The proof can be found in [4, see Lemma 2].

Lemma 1. *Let k and ℓ be integers with $k > \ell \geq 0$. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$, $p | b_j$ for all $j \in \{0, 1, \ldots, n-\ell-1\}$, and the right-most edge of the Newton polygon for $g(x)$ with respect to p has slope $< 1/k$. Then for any integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_{n-j} b_j x^j$ cannot have a factor with degree in the interval $[\ell+1, k]$.*

We take $g(x) = z_n(x)$. Following the proofs of Lemmas 4 and 5 from [4], we note that the slope of the right-most edge of the Newton polygon of $z_n(x)$ with respect to p is

$$\max_{1 \leq u \leq n} \left\{ \frac{1}{u} \left(\nu(u!) + \nu\left(\frac{(2n)!}{(2n-u)!} \right) - \nu\left(\frac{n!}{(n-u)!} \right) \right) \right\}.$$

We will be interested in using this expression for the slope when applying Lemma 1 to obtain information about the degrees of the factors of $f(x)$. First, we indicate the information obtained from Lemma 1 in [4] (cf. Lemmas 4 and 5 there).

Lemma 2. *Let n be a positive integer. Suppose that p is a prime, that k and r are positive integers, and that ℓ is an integer in $[0, k-1]$ for which*

(i) $p^r || (n-\ell)$,

(ii) $p \geq 2\ell + 1$,

and

(iii) $\dfrac{\log(2n)}{p^r \log p} + \dfrac{1}{p-1} \leq \dfrac{1}{k}$.

Then $f(x)$ cannot have a factor with degree $\in [\ell+1, k]$.

Lemma 3. *Let n be a positive integer. Suppose that p is a prime, that k and r are positive integers, and that ℓ is an integer in $[-k, -1]$ for which*

(i) $p^r || (n-\ell)$,

(ii) $p \geq 2|\ell| + 1$,

and

(iii) $\max\left\{\dfrac{1}{p-2|\ell|+1}, \dfrac{\log(2n)}{(p^r-2|\ell|+1)\log p}\right\} + \dfrac{1}{p-1} \le \dfrac{1}{k}$.

Then $f(x)$ cannot have a factor with degree $\in [|\ell|, k]$.

In [4], Lemma 3 was only used to show that for n large $f(x)$ cannot have a factor of degree 1, 2, 3, or 4; furthermore, only the case $\ell = -1$ was used. In this paper, we will take advantage of the full range of values of ℓ given in the above two lemmas. The condition given in Lemma 3 (iii) is somewhat awkward and can be improved when $r = 1$. The purpose of our next lemma is to obtain such an improvement and at the same time uniformize our use of Lemmas 2 and 3.

Lemma 4. *Let n be a positive integer. Suppose that p is an odd prime, that k is an integer > 1, and that $\ell \in \{-k, -(k-1), \ldots, -1, 0, 1, \ldots, k-2, k-1\}$. Suppose further that*

(i) $p|(n-\ell)$,

(ii) $p \ge 2k+1$,

and

(iii) $\dfrac{\log(2n)}{(p-2k+1)\log p} < \dfrac{1}{k}$.

Then $f(x)$ cannot have a factor of degree k.

Proof. In order to prove Lemma 4, we consider the cases $\ell \ge 0$ and $\ell < 0$ separately. For $\ell \ge 0$, we show the above lemma is in fact a consequence of Lemma 2. To do this we need only show that the inequality in Lemma 4 (iii) implies the inequality in Lemma 2 (iii); more precisely, we establish

$$\frac{\log(2n)}{(p-2k+1)\log p} < \frac{1}{k} \quad \Longrightarrow \quad \frac{\log(2n)}{p\log p} + \frac{1}{p-1} < \frac{1}{k} \tag{2}$$

from which Lemma 2 (iii) is an easy consequence. By a direct computation, we have

$$\frac{1}{pk}\left(\frac{1}{k} - \frac{1}{p-1}\right)^{-1} = \frac{p-1}{p(p-k-1)} \le \frac{1}{p-2k+1}.$$

The inequality in Lemma 4 (iii) implies

$$\frac{1}{pk}\left(\frac{1}{k} - \frac{1}{p-1}\right)^{-1}\frac{\log(2n)}{\log p} \le \frac{\log(2n)}{(p-2k+1)\log p} < \frac{1}{k},$$

and the implication in (2) easily follows.

Now, consider the case $\ell < 0$. We modify the argument given for Lemma 5 in [4] as follows. Observe that $z_n(x) = \sum_{m=0}^{n} c_m x^{n-m}$ where $c_0 = 1$ and

$$c_m = \binom{n}{m}(n+1)\cdots(n+m) \qquad \text{for } m \ge 1.$$

Hence, $p|c_m$ for $m \in \{|\ell|, |\ell|+1, \dots, n\}$. We apply Lemma 1 and make use of the description of the slope of the right-most edge of the Newton polygon of $z_n(x)$ with respect to p. Thus, we want to show that for each $u \in \{1, 2, \dots, n\}$,

$$\frac{1}{u}\left(\nu(u!) + \nu\left(\frac{(2n)!}{(2n-u)!}\right) - \nu\left(\frac{n!}{(n-u)!}\right)\right) < \frac{1}{k}. \tag{3}$$

Fix such a u, and define $a(x,j)$ as the number of multiples of p^j in $(x-u, x]$. Then

$$\nu\left(\frac{(2n)!}{(2n-u)!}\right) - \nu\left(\frac{n!}{(n-u)!}\right) = \sum_{j=1}^{\infty}\big(a(2n,j) - a(n,j)\big).$$

The upper limit in the summation may be replaced by $[\log(2n)/\log p]$ since $a(2n,j) = a(n,j) = 0$ when $p^j > 2n$. Furthermore,

$$a(x,j) = \left[\frac{x}{p^j}\right] - \left[\frac{x-u}{p^j}\right] = \frac{u}{p^j} + \theta(x,j)$$

where $|\theta(x,j)| < 1$. It follows that $a(2n,j) - a(n,j)$ is an integer with absolute value < 2. Therefore, $|a(2n,j) - a(n,j)| \le 1$. Hence,

$$\nu\left(\frac{(2n)!}{(2n-u)!}\right) - \nu\left(\frac{n!}{(n-u)!}\right) \le \sum_{1 \le j \le [\log(2n)/\log p]} \big|a(2n,j) - a(n,j)\big| \le \frac{\log(2n)}{\log p}.$$

Also,

$$\nu(u!) = \sum_{j=1}^{\infty}\left[\frac{u}{p^j}\right] \le \sum_{j=1}^{\infty}\frac{u}{p^j} = \frac{u}{p-1}.$$

Suppose first that $u \ge p$. Then

$$\frac{1}{u}\left(\nu(u!) + \nu\left(\frac{(2n)!}{(2n-u)!}\right) - \nu\left(\frac{n!}{(n-u)!}\right)\right) \le \frac{1}{p-1} + \frac{\log(2n)}{u\log p} \le \frac{1}{p-1} + \frac{\log(2n)}{p\log p}.$$

From (2), it follows that Lemma 4 (iii) implies this last expression is $< 1/k$. We show next that (3) holds when $u < p$ so that we may deduce that the right-most edge of the Newton polygon of $z_n(x)$ has slope $< 1/k$ and apply Lemma 1.

Suppose $u < p$. Then $\nu(u!) = 0$. If $1 \le u \le p + 2\ell$, then $n - \ell - p < n - u + 1$ and $2n - 2\ell - p < 2n - u + 1$ so that there are no multiples of p in each of $(n-u, n]$ and $(2n-u, 2n]$. In this case,

$$\frac{1}{u}\left(\nu(u!) + \nu\left(\frac{(2n)!}{(2n-u)!}\right) - \nu\left(\frac{n!}{(n-u)!}\right)\right) = 0.$$

If $u \ge p + 2\ell + 1$, then $u \ge p - 2k + 1$ so that

$$\frac{1}{u}\left(\nu(u!) + \nu\left(\frac{(2n)!}{(2n-u)!}\right) - \nu\left(\frac{n!}{(n-u)!}\right)\right) \le \frac{1}{u} \sum_{1 \le j \le [\log(2n)/\log p]} \left|a(2n,j) - a(n,j)\right|$$
$$\le \frac{\log(2n)}{u \log p} \le \frac{\log(2n)}{(p-2k+1)\log p}.$$

We deduce from Lemma 4 (iii) that this last expression is $< 1/k$.

Combining the above, we obtain that (3) holds for each $u \in \{1, 2, \ldots, n\}$ and, hence, the slope of the right-most edge of the Newton polygon of $z_n(x)$ is $< 1/k$. Lemma 1 implies that $f(x)$ cannot have a factor of degree k, completing the proof. ■

3. The Proof of the Theorem

In this section, we let $f(x)$ denote the polynomial given in (1). We assume that $f(x)$ has a factor of degree $k \in [1, n/2]$ and establish the theorem by obtaining a contradiction. We break the argument up into different cases depending on the size of k. For these different cases, we will establish the theorem for $n \ge 2479$. The final case we consider is the irreducibility of $f(x)$ when $n < 2479$. Throughout our arguments we will make use of the following explicit analytic estimates of Rosser and Schoenfeld [10, formulas (3.6), (3.14), (3.15), and (3.16)].

Lemma 5. *Let $\pi(x)$ denote the number of primes $p \le x$, and let $\vartheta(x) = \sum_{p \le x} \log p$. Then*

$$\pi(x) < 1.256 \frac{x}{\log x} \quad \text{for all } x > 1,$$

$$\vartheta(x) < x\left(1 + \frac{1}{2\log x}\right) \quad \text{for all } x > 1, \quad \text{and} \quad \vartheta(x) > \begin{cases} x\left(1 - \dfrac{1}{\log x}\right) & \text{for } x \ge 41 \\ x\left(1 - \dfrac{1}{2\log x}\right) & \text{for } x \ge 563. \end{cases}$$

CASE 1: $\dfrac{n}{100} \le k \le \dfrac{n}{2}$ and $n \ge 2479$.

Lemma 6. *For $x \ge 2479$, there is a prime in the interval $(x, 1.01x]$.*

Proof. Let $\vartheta(x) = \sum_{p \le x} \log p$. There is a prime in $(x, 1.01x]$ provided $\vartheta(1.01x) > \vartheta(x)$. We deduce from Lemma 5 that, for $x \ge 563$, there is a prime in $(x, 1.01x]$ provided

$$1.01x\left(1 - \frac{1}{2\log(1.01x)}\right) > x\left(1 + \frac{1}{2\log x}\right).$$

One easily checks that this holds if $\log x \ge 101$. Since $e^{101} < 10^{44}$, we are through if $x > 10^{44}$.

We are left with verifying the theorem for all $x \in [2479, 10^{44}]$. We define a sequence recursively as follows. Let $x_0 = 2479$. For $n \geq 1$, define x_n as the largest prime $< 1.01x_{n-1}$. Suppose that $x_{n+1} > x_n$. It follows that if $x \in [x_n, x_{n+1})$, then there is a prime $p \in (x, 1.01x]$ (the prime x_{n+1} will be such a p). We computed x_n for $1 \leq n \leq 10^4$ using both MAPLE (Version V, Release 4) and PARI (Version 1.39.12). Each of these contains a subroutine called "isprime" which, in both cases, is a pseudo-primality test. We avoided use of this subroutine and instead used the factoring routines in these symbolic packages. We compared the values of x_{1000j} for $1 \leq j \leq 10$. The results were identical with

$$x_{10000} = 30160992555276892299261579269275542298335694063 > 10^{47}.$$

This establishes the lemma. ■

By Lemma 6, there is a prime $p = n + \ell + 1$ where $0 \leq \ell < n/100$. As in the proof of Lemma 4, we use that $z_n(x) = \sum_{m=0}^{n} c_m x^{n-m}$ where $c_0 = 1$ and

$$c_m = \binom{n}{m}(n+1)(n+2)\cdots(n+m) \quad \text{for } m \geq 1.$$

Thus, $p|c_m$ for $m \in \{\ell+1, \ell+2, \ldots, n\}$. The endpoints of the right-most edge of the Newton polygon of $z_n(x)$ with respect to p are $(\ell, 0)$ and $(n, 1)$. This edge, therefore, has slope $1/(n-\ell)$. By Lemma 1, $f(x)$ cannot have a factor with degree in $(\ell, n-\ell)$. Hence, k cannot be in the range given for this case.

CASE 2: $n^{1/2} \leq k < \dfrac{n}{100}$.

Observe that the conditions in this case imply $n > 10^4$ and $k > 100$. To show no factor can be of degree k, we show that there is a prime $p > 3k > (2n)^{1/2}$ that divides $n(n-1)\cdots(n-k+1)$. Once we have done this, Lemma 2 implies we are through since

$$\frac{\log(2n)}{p^r \log p} + \frac{1}{p-1} < \frac{\log(2n)}{p\log((2n)^{1/2})} + \frac{1}{3k} < \frac{2}{3k} + \frac{1}{3k} = \frac{1}{k}.$$

Since $k > 41$, we deduce from Lemma 5 that

$$\prod_{k<p\leq 3k} p = \exp\left(\vartheta(3k) - \vartheta(k)\right) \leq \exp\left(2k + \frac{3k}{2\log(3k)} + \frac{k}{\log k}\right) \leq \exp\left(2k + \frac{2.5k}{\log k}\right).$$

We make use of an idea of Erdős [1] (also described by Tijdeman in [12]). For each prime $p \leq k$, we consider a number among $n, n-1, \ldots, n-k+1$ which is divisible by p^e where $e = e(p)$ is as large as possible. We dispose of these, and let S denote the set of numbers that remain. For every prime p, let N_p denote the exponent in the largest power of p dividing $\prod_{m\in S} p$. For $p \leq k$, we obtain $N_p \leq [k/p] + [k/p^2] + \cdots$. For $k < p \leq 3k$, we use that $k \geq n^{1/2}$ to deduce that $N_p \leq 1$. Therefore,

$$\prod_{p>3k} p^{N_p} = \frac{\prod_{m\in S} m}{\left(\prod_{p\leq k} p^{N_p}\right)\left(\prod_{k<p\leq 3k} p^{N_p}\right)} \geq (n-k+1)^{|S|}\left(k!\exp\left(2k + \frac{2.5k}{\log k}\right)\right)^{-1}.$$

To complete this case, it suffices to show this last expression is > 1. We use that $n-k+1 > 99k$, $|S| > k - 1.256k/\log k$ (by Lemma 5), and $k! < ((k+1)/e)^{k+1}$ for $k \geq 6$. Taking logarithms, we want

$$k\log k + \left((\log 99) - 1.256\right)k - (\log 99)(1.256)\frac{k}{\log k} > (k+1)\log(k+1) + (k-1) + 2.5\frac{k}{\log k}.$$

Since $k > 100$, we have $\log k \geq \log 100 > 4$. Also, $(\log x)/x$ is decreasing for $x > e$ so that $\log k < k(\log 100)/100 < (0.05)k$. Finally, we use that $\log(k+1) \leq (\log k) + (1/k)$. A direct computation shows the above inequality holds.

CASE 3: $100 \leq k < n^{\frac{1}{2}}$.

We obtain a contradiction here by modifying the approach used in the previous case. In Cases 3, 4, and 5, we use one and the same construction. We give details here though not all aspects of our discussion will be needed for Case 3. Define $p_{min} = p_{min}(k)$ as the least prime $\geq 2k + \sqrt{2k^2 - 2k + 1}$. Hence,

$$p \geq p_{min} \quad \Longrightarrow \quad \frac{1}{p - 2k + 1} + \frac{1}{p - 1} \leq \frac{1}{k}.$$

For each prime p, define $r = r(p)$ so that $p^{r(p)} || (n+k)(n+k-1)\cdots(n-k+1)$. Define $P = P(k) = \prod_{p \geq p_{min}} p^{r(p)}$. We estimate P in two different ways. First we get a lower bound. Using the idea of Erdős [1], for each prime $p < p_{min}$, we consider a number among $n+k, n+k-1, \ldots, n-k+1$ which is divisible by p^e where $e = e(p)$ is as large as possible. We dispose of these, and let S be the set of numbers that remain. For every prime p, let N_p denote the exponent of the largest power of p dividing $\prod_{m \in S} m$. Obviously $r(p) \geq N_p$ for each prime p and $\prod_{m \in S} m = \prod_p p^{N_p}$. As before, for $p < p_{min}$ we obtain $N_p \leq [2k/p] + [2k/p^2] + \cdots$ so that $\prod_{p < p_{min}} p^{N_p} \leq (2k)!$. Since $P \geq \prod_{p \geq p_{min}} p^{N_p}$, we get

$$P > \frac{(n-k)^{2k - \pi(p_{min} - 1)}}{(2k)!}. \tag{4}$$

Next, we get an upper bound for P. Note that when $p \geq p_{min} > 2k$ at most one of the numbers $n+k, n+k-1, \ldots, n-k+1$ is divisible by p. Let p_0 be a prime $\geq p_{min}$. We estimate $P_1(p_0) = \prod_{p \geq p_0, r(p) > 1} p^{r(p)}$ (where we define the product to be 1 if $r(p) \leq 1$ for each $p \geq p_0$). Define $P_1 = P_1(p_{min})$. Let p, if it exists, be a prime $\geq p_0$ with $r(p) > 1$. From Lemma 2 and Lemma 3, we get

$$\frac{\log(2n)}{(p^r - 2k + 1)\log p} + \frac{1}{p-1} > \frac{1}{k}.$$

Defining

$$\alpha_k = \alpha_k(p_0) = \frac{1}{k} - \frac{1}{p_0 - 1} \quad \text{and} \quad \beta_k = \beta_k(p_0) = \alpha_k(p_0^2 - 2k + 1),$$

we deduce that

$$\alpha_k < \frac{\log(2n)}{(p^r - 2k + 1)\log p} \quad \text{and} \quad \beta_k < \frac{\log(2n)}{\log p}.$$

These inequalities hold provided $p \geq p_0$ and $r(p) > 1$ as in the product defining $P_1(p_0)$. If $\beta_k \geq \log(2n)/\log p_0$, then it follows that no such p exists. In other words,

$$n \leq \frac{p_0^{\beta_k}}{2} \quad \Longrightarrow \quad P_1(p_0) = 1. \tag{5}$$

The above inequalities also imply for $r > 1$ that

$$2k - 1 + \frac{\log(2n)}{\alpha_k \log p} > p^r \quad \text{and} \quad \frac{\gamma_k \log(2n)}{\log p} > p^r \text{ where } \gamma_k = \gamma_k(p_0) = \frac{1}{\alpha_k} + \frac{2k-1}{\beta_k}. \tag{6}$$

Observe that when $k \geq 4$, we have $2k + \sqrt{2k^2 - 2k + 1} \geq 3k + 1$. Also, $p_{min}(3) = 11$ and $p_{min}(2) = 7$. It follows that for $k > 1$, we have

$$p_{min} \geq 3k + 1, \quad \alpha_k \geq \frac{2}{3k}, \quad \beta_k > 6k + \frac{8}{3}, \quad \text{and } \gamma_k < \frac{3k}{2} + \frac{1}{3}.$$

Set $z = z(k, p_0) = \gamma_k \log(2n)/\log p_0$. Then $z > p^{r(p)} > 1$ and $\sqrt{z} > p$ for each p in the product. In particular, if such a p exists (that is if $P_1(p_0) > 1$), then $\pi(\sqrt{z}) > \pi(p_0 - 1)$ and $P_1(p_0) \leq z^{\pi(\sqrt{z}) - \pi(p_0 - 1)}$. Lemma 5 implies

$$z^{\pi(\sqrt{z})} < \exp(2.512\sqrt{z}) = (2n)^{\frac{2.512\sqrt{z}}{\log(2n)}}.$$

Since $\beta_k < \log(2n)/\log p_0$, we have

$$\frac{\sqrt{z}}{\log(2n)} < \frac{1}{\log p_0}\sqrt{\frac{\gamma_k}{\beta_k}}.$$

It follows that if there is a prime $p \geq p_0$ for which $r(p) > 1$, then

$$P_1(p_0) \leq (2n)^{\delta_k}/z^{\pi(p_0 - 1)} \qquad \text{where } \delta_k = \frac{2.512}{\log p_0}\sqrt{\frac{\gamma_k}{\beta_k}}. \tag{7}$$

Also, $\sqrt{z} > p_0$ implies

$$z^{\pi(p_0 - 1)} \geq z^{\pi(p_{min} - 1)} \geq p_0^{2\pi(p_{min} - 1)}.$$

Define

$$B = \max\left\{1, \frac{(2n)^{\delta_k}}{p_0^{2\pi(p_{min} - 1)}}\right\}.$$

Then we obtain

$$P_1(p_0) \leq B. \tag{8}$$

Now, we estimate $P_2 = \prod_{p \geq p_{min}, r(p)=1} p$ (where we define the product to be 1 if $r(p) \neq 1$ for each $p \geq p_{min}$). If $P \neq 1$, we let p_{max} denote the largest prime in the product P. Then $P_2 \leq \exp(\vartheta(p_{max}) - \vartheta(p_{min} - 1))$. For $k > 1$, Lemma 4 implies

$$2k - 1 + \frac{k \log(2n)}{\log p} \geq p \tag{9}$$

for each prime appearing in the product P. In particular, (9) holds with $p = p_{max}$ (if $P \neq 1$). Lemma 5 implies $\vartheta(p_{max}) \leq c_1(k) + c_2(k) \log n$ where

$$c_1(k) = \left(1 + \frac{1}{2 \log p_{max}}\right) \left(2k - 1 + \frac{k \log 2}{\log p_{max}}\right)$$

and

$$c_2(k) = \left(1 + \frac{1}{2 \log p_{max}}\right) \frac{k}{\log p_{max}}.$$

Thus, if $P \neq 1$, then

$$P_2 \leq \frac{e^{c_1(k)} n^{c_2(k)}}{\prod_{p < p_{min}} p}. \tag{10}$$

Observe that if $P = 1$, then $P_2 = 1$ so that in general P_2 is bounded above by the maximum of 1 and the expression on the right of (10). Also, note that (4) through (10) hold true for all $n > 2k > 2$. Furthermore, (4) through (8) hold when $k = 1$.

Now, we use these estimates to complete Case 3. We have $n - k > n(1 - \frac{1}{k})$. Also, since $(1 - \frac{1}{k})^k$ is increasing for $k \geq 1$, we obtain $(1 - \frac{1}{k})^k \geq (0.99)^{100}$. Lemma 5 implies

$$\pi(p_{min} - 1) \leq \pi((2 + \sqrt{2})k) < 1.256 \frac{(2 + \sqrt{2})k}{\log((2 + \sqrt{2}) \times 100)} < 0.74k.$$

We use that for $m \geq 50$, the inequality $m! < (m/e)^{m+1}$ holds. Since $k > 25$ and $n > k^2$, we deduce that $(2k)! < (2k/e)(2/e)^{2k} n^k$. Using (4) we obtain

$$P > \frac{1}{3k} \left(\frac{e}{2}\right)^{2k} n^{0.26k} > \frac{n^{0.26k}}{3k}.$$

Recall that for $k > 1$, $\beta_k > 6k + (8/3)$ and $\gamma_k < (3k/2) + (1/3)$. Thus, $\gamma_k / \beta_k < 1/4$. Also, $p_{min}(100) = 347$ and $\pi(346) = 68$. From (8), we deduce that

$$P_1 < n^{0.3}.$$

Using $p_{max}(k) \geq p_{min}(k) \geq p_{min}(100) = 347$ we get $c_1(k) < 2.3k$ and $c_2(k) < 0.2k$. For $k \geq 100$, we obtain from Lemma 5 that

$$\log\left(\prod_{p < p_{min}} p\right) = \vartheta(p_{min} - 1) \geq \vartheta(3k) > \left(1 - \frac{1}{\log 300}\right) 3k > 2.3k.$$

Therefore, (10) implies $P_2 \le n^{0.2k}$.

Since $P = P_1 P_2$, combining the above, we deduce

$$\frac{n^{0.26k}}{3k} < n^{0.3} n^{0.2k}.$$

In other words, $n^{0.06k-0.3} < 3k$. Since $\sqrt{n} > k \ge 100$, we obtain $n^{0.06k-0.3} \ge n^5 \ge 3k$. The previous inequality therefore cannot hold, and we obtain a contradiction. Thus, $f(x)$ cannot have a factor of degree k in this case.

CASE 4: $3 \le k < 100$ and $n \ge 1800$.

We proceed in the same way as we did in the previous case, the only difference being that we compute the quantities depending on k but not on n instead of estimating them. Our computations were done with the use of MAPLE V, Release 4. In particular, we calculated $p_{min} = p_{min}(k)$ and $\pi(p_{min} - 1)$ precisely for $3 \le k < 100$ to determine that the right-hand side of (4) is > 1 when $n = 805$. It follows that $P > 1$ whenever $n > 804$ and $3 \le k < 100$. Since in fact $n \ge 1800$, we also have $n - k = n(1 - k/n) \ge n(1 - k/1800)$. From (4), we obtain

$$P > \frac{(1 - \frac{k}{1800})^s n^s}{(2k)!} \quad \text{where} \quad s = 2k - \pi(p_{min} - 1).$$

We consider first the possibility that $P_1 > 1$ and take $p_0 = p_{min}$. Then (8) implies

$$P_1 \le \frac{(2n)^{\delta_k}}{p_{min}^{2\pi(p_{min}-1)}}.$$

Using (10) and $P = P_1 P_2$, we obtain

$$\frac{(1 - \frac{k}{1800})^s n^s}{(2k)!} < \frac{(2n)^{\delta_k}}{p_{min}^{2\pi(p_{min}-1)}} \times \frac{e^{c_1(k)} n^{c_2(k)}}{\prod_{p<p_{min}} p}.$$

Taking logarithms, we deduce that $e_1(k) \log n + e_2(k) < 0$ where

$$e_1(k) = s - c_2(k) - \delta_k$$

and

$$e_2(k) = \vartheta(p_{min}-1) + s \log\left(1 - \frac{k}{1800}\right) + 2\pi(p_{min}-1)\log(p_{min}) - \log((2k)!) - \delta_k(\log 2) - c_1(k).$$

We find trivial upper bounds for $c_1(k)$ and $c_2(k)$ by using the estimate $p_{max}(k) \ge p_{min}(k)$. Observe that with this estimate for $p_{max}(k)$, we obtain lower bounds for $e_1(k)$ and $e_2(k)$ that depend only on k. Direct computations show that $e_1(k) > 0$ and $e_2(k) > 0$ for each

$k \in [3, 100)$. Thus, in the case that $P_1 > 1$, we deduce that $f(x)$ has no factors of degree $k \in [3, 100)$.

We now consider the possibility that $P_1 = 1$. Since $P = P_2$, we obtain $e_1'(k)\log n + e_2'(k) < 0$ where $e_1'(k) = e_1(k) + \delta_k > e_1(k) > 0$ and $e_2'(k) = e_2(k) - 2\pi(p_{min} - 1)\log(p_{min}) + \delta_k(\log 2)$. Direct computations give here that

$$e_1'(k)\log n + e_2'(k) \geq e_1'(k)\log(1800) + e_2'(k) > 7.4e_1'(k) + e_2'(k) > 0 \quad \text{for } 8 \leq k < 100.$$

Therefore, $f(x)$ has no factors of degree $k \in [8, 100)$. The cases $k = 3$, 4, 5, 6, and 7 require some additional work.

For $3 \leq k \leq 7$, this same argument works provided $n > N$ where $N = N(k) = \exp(-e_2'/e_1')$ (since then $e_1'(k)\log n + e_2'(k) > 0$). Further work is required to handle $n \in [1800, N]$. For such n, we use (9) to obtain

$$2k - 1 + \frac{k\log(2N)}{\log p} - p \geq 0$$

for each prime p dividing P_2. As a function of p, the left-hand side of the above inequality is decreasing. We determine the least prime $q = q(k)$ such that the inequality does not hold if p is replaced by q. Then $p_{max} < q$. Using the definition of p_{min}, we obtain

$$k = 7 \implies 29 = p_{min} \leq p_{max} \leq 31, \quad k = 6 \implies 23 = p_{min} \leq p_{max} \leq 29,$$

$$k = 5 \implies 17 = p_{min} \leq p_{max} \leq 23, \quad k = 4 \implies 13 = p_{min} \leq p_{max} \leq 19,$$

and

$$k = 3 \implies 11 = p_{min} \leq p_{max} \leq 19.$$

Since in this case $P = P_2$ is at most the product of the primes from p_{min} to p_{max}, we obtain an upper bound on $P = P(k)$ for each $k \in [3, 7]$. Specifically, we have

$$P(7) \leq 29 \times 31 = 899, \quad P(6) \leq 23 \times 29 = 667, \quad P(5) \leq 17 \times 19 \times 23 = 7429,$$

$$P(4) \leq 13 \times 17 \times 19 = 4199, \quad \text{and} \quad P(3) \leq 11 \times 13 \times 17 \times 19 = 46189.$$

On the other hand, using (4) with $n \geq 1800$, we obtain the lower bounds

$$P(7) \geq 212565, \; P(6) \geq 21624, \; P(5) \geq 2860847, \; P(4) \geq 143680, \quad \text{and} \quad P(3) \geq 4485.$$

For $4 \leq k \leq 7$, the lower bounds on $P(k)$ exceed the upper bounds and we deduce again that $f(x)$ cannot have a factor of degree k. For $k = 3$, we iterate the above procedure a second time. Using (4), we obtain that $P(3) \geq 46192$ if $n \geq 5770$. Since we have already determined that $P(3) \leq 46189$, we obtain $n \leq 5769$. Replacing $N(3)$ with 5769 above, we deduce that $p_{max}(3) \leq 13$ so that $P(3) \leq 11 \times 13 = 143$. This contradicts our lower bound for $P(3)$ and we conclude that $f(x)$ cannot have a factor of degree 3.

CASE 5: $k = 1$ or 2 and $n \geq 1614$.

First, assume that $f(x)$ has a factor of degree 2 (i.e., $k = 2$). In this case, $p_{min} = p_{min}(2) = 7$. Since among any four consecutive integers there is exactly one divisible by 4, at most one divisible by 9, and at most one divisible 5, we have $P = \prod_{p \geq 7} p^r \geq (n-1)/6$. We need to refine somewhat our estimates for P_1 and P_2.

We consider first the possibility that $p_{max} \geq 23$, that is that the largest prime divisor of $(n+2)(n+1)n(n-1)$ is ≥ 23. Then Lemma 4 implies $n \geq (1/2)23^{10}$. By using (10) and $p_{max} \geq 23$ we obtain $P_2 \leq 2 \times n^{0.74}$.

To estimate P_1 we first show that $r(p) \leq 1$ for all primes $p \geq 11$. Assume otherwise so that $r(p) > 1$ for some prime $p \geq 11$. From (6) (with $p_0 = p_{min} = 7$) we obtain $7^{r(7)} < 2\log(2n)$. From (7), we see that $P_1(11) < 2n^{0.245}/z^4$ where $z = z(2, 11) = \gamma_2(11)\log(2n)/\log(11) > \log(2n)$. Recall also from the arguments there that $\sqrt{z} > p$. Hence, $z \geq 11^2 = 121$. We obtain

$$P_1 = P_1(7) \leq 7^{r(7)} \times P_1(11) < 2n^{0.245} \frac{2\log(2n)}{\log(2n) \times 121^3} < 10^{-5} n^{0.245}.$$

We deduce that

$$n - 1 \leq 6P = 6P_1P_2 < 6 \times 2 \times 10^{-5} n^{0.985} < \frac{n^{0.985}}{10}.$$

By dividing through by $n^{0.985}$, it is easy to see that the above inequalities cannot hold for $n \geq 2$. Since $n \geq (1/2)23^{10}$, we obtain a contradiction.

Thus, we must have $r(p) \leq 1$ for all primes $p \geq 11$. But then we obtain

$$P_1 \leq 7^{r(7)} < 2\log(2n) \quad \Longrightarrow \quad n - 1 \leq 6P = 6P_1P_2 < 24\log(2n)n^{0.74}.$$

By dividing through by $n^{0.74}$ and rearranging, we obtain the function $w(n) = n^{0.26} - n^{-0.74} - 24\log(2n)$. It is easy to deduce that $w(n)$ is increasing for $n > 10^8$ and that $w(10^{11}) > 0$. Since we are interested in $n \geq (1/2)23^{10} > 10^{11}$, we obtain again a contradiction. We deduce that $p_{max} \leq 19$.

Since $p_{min} = 7$ and $p_{max} \leq 19$, there are at most five primes p in the product P for which $r(p) > 0$. From Lemma 2 and Lemma 3, we obtain $p^r < 3 + (3\log(2n)/\log(7))$ for any prime $p \geq 7$ so that

$$n - 1 \leq 6P = 6\prod_{p \geq 7} p^r \leq 6\left(3 + \frac{3\log(2n)}{\log 7}\right)^5. \tag{11}$$

Dividing through by n and substituting $x = n^{1/5}$, we are led to considering the function

$$w(x) = 1 - \frac{1}{x^5} - 6\left(\frac{3}{x} + \frac{15\log x + 3\log 2}{x\log 7}\right)^5.$$

It is easy to see that $w(x)$ is increasing for $x > e$. Since $w(50) > 0$, we deduce that (11) cannot hold if $n > 50^5$. Therefore, $f(x)$ has no factors of degree 2 for such values of n. Let $n \le 50^5$. Note that (5) implies $P_1 = 1$ for $n \le (1/2)7^{(46/3)}$. In particular, $P_1 = 1$ for $n \le 50^5$. Therefore, in this case,

$$n - 1 \le 6P = 6P_2 \le 6 \times 7 \times 11 \times 13 \times 17 \times 19 < 10^7.$$

Now, from (9) we deduce that $p_{max} \le 13$. Hence, $n - 1 \le P \le 6 \times 7 \times 11 \times 13$ so that $n \le 6007$. One more application of (9) gives $p_{max} \le 7$ which implies $n - 1 \le 6 \times 7$. Thus, $f(x)$ has no irreducible factors of degree 2 when $n > 43$.

The case $k = 1$ is similar to the case $k = 2$. With $k = 1$, we have $p_{min} = 3$. At least one of n and $n+1$ is odd so that the definition of P implies $P = \prod_{p \ge 3} p^r \ge n$. We consider first the possibility that the largest prime divisor of $n(n+1)$ is ≥ 13 (in other words, $p_{max} \ge 13$). Lemma 2 and Lemma 3 imply $n \ge (1/2)13^{11} > 10^{11}$. We wish to apply (10) except our argument for (10) made use of (9) which holds only for $k > 1$. By using Lemma 2 and Lemma 3 again we obtain for each $p > 2$ that $p < 2 + \log(2n)/\log p$, and we replace our use of (9) (to derive (10)) with this inequality. For $k = 1$ we deduce that (10) holds with

$$c_1(1) = \left(1 + \frac{1}{2\log p_{max}}\right)\left(2 + \frac{\log 2}{\log p_{max}}\right)$$

and $c_2(1)$ as defined earlier. Using $p_{max} \ge 13$ in (10), we deduce that $P_2 < 8\sqrt{n}$.

To estimate P_1 we show first that $r(p) \le 1$ for all primes $p \ge 5$. Assume $r(p) > 1$ for some prime $p \ge 5$. From (6) we obtain $3^r < 2.5\log(2n)$. Using (7), we deduce that $P_1(5) < 2 \times n^{0.45}/z^2$ where $z = z(1,5) = \gamma_1(5)\log(2n)/\log 5 \ge 0.8\log(2n)$. Also, $\sqrt{z} > p$ implies $z \ge 25$. We obtain

$$P_1 = P_1(3) = 3^{r(3)} \times P_1(5) < 2 \times n^{0.45}\frac{2.5\log(2n)}{0.8\log(2n) \times 25} = \frac{n^{0.45}}{4}.$$

We deduce that $n \le P = P_1P_2 < 2 \times n^{0.95}$. This does not hold for $n \ge 2^{20}$. Since $n > 10^{11} > 2^{20}$, we arrive at a contradiction. Therefore, $r(p) \le 1$ for all primes ≥ 5. Now, we obtain

$$P_1 \le 3^r < 2.5\log(2n) \quad \text{and} \quad n \le P < 2.5\log(2n) \times 8\sqrt{n} = 20\log(2n)\sqrt{n}.$$

The function $w(x) = 20\log(2x)/\sqrt{x}$ is easily seen to be decreasing for $x > 4$. Also, $w(10^5) < 1$. It follows that the inequality on n above does not hold for $n \ge 10^5$. Since $n > 10^{11}$, we deduce that the largest prime divisor of $n(n+1)$ is < 13.

It remains to consider the case when $p_{max} \le 11$. Lemma 2 and Lemma 3 imply

$$p^r < 1 + \frac{2\log(2n)}{\log p} \le 1 + \frac{2\log(2n)}{\log 3} \qquad \text{for each prime dividing } P. \tag{12}$$

Therefore,

$$n \le P = \prod_{p \ge 3} p^r < \left(1 + \frac{2\log(2n)}{\log 3}\right)^4.$$

By considering $w(x) = x^{-1/4} + (2\log(2x)/(x^{1/4}\log(3))$ in a manner similar to before, we deduce that the above inequality does not hold for $n \geq 500000$. Using the inequality $p < 2 + \log(2n)/\log p$ mentioned above on primes p dividing P and $n < 500000$, we obtain $p_{max} \leq 7$. From (12), we now see that

$$n \leq P \leq \left(1 + \frac{2\log(2n)}{\log 3}\right)\left(1 + \frac{2\log(2n)}{\log 5}\right)\left(1 + \frac{2\log(2n)}{\log 7}\right).$$

For each $p \geq 2$, the function $w(x) = x^{-1/3} + (2\log(2x)/(x^{1/3}\log(p))$ is decreasing for $x \geq 21 > e^3$. We deduce that the inequality above does not hold if $n \geq 1614$. Thus, we conclude that $f(x)$ does not have a linear factor for $n \geq 1614$.

CASE 6: $n \leq 2479$.

We suppose as we may that $n \geq 2$. We verify that $f(x)$ is irreducible whenever $2 \leq n \leq 2479$ as follows. Fix such an n. We use Lemma 4 to prove that $f(x)$ cannot have a factor of degree $k \in [2, n/2]$, and then we address the possibility that $f(x)$ has a linear factor. We make use of three primes p_1, p_2, and p_3. We define p_1 as the smallest prime $\geq n$. We define p_2 as the largest prime divisor of $(n-1)n(n+1)(n+2)$. We let t be the least positive integer satisfying

$$(p_2 - 2t + 1)\log p_2 \leq t\log(2n).$$

Define p_3 as the largest prime divisor of $\prod_{j=-t+1}^{t}(n+j)$. The prime p_1 is used in a manner similar to Case 1. We deduce that $f(x)$ cannot have a factor of degree $k \in (\alpha, n/2]$ where $\alpha = p_1 - n$. By the definition of t, for each $k < t$, we have $(p_2 - 2k + 1)\log p_2 > k\log(2n)$. This implies $p_2 > 2k$ for each such k. Since also $p_2|(n-1)n(n+1)(n+2)$, the conditions in Lemma 4 hold, and we deduce that $f(x)$ cannot have a factor of degree $k \in [2, t)$. If $t > \alpha$, then we can conclude $f(x)$ has no factor of degree $k \in [2, n/2]$. If $t \leq \alpha$, we make use of p_3. We checked computationally using MAPLE, and for each $n \in [2, 2479]$ (and somewhat beyond) the inequalities

$$p_3 > 2k \quad \text{and} \quad (p_3 - 2k + 1)\log p_3 > k\log(2n)$$

held for $t \leq k \leq \alpha$; indeed, the computation was somewhat simple as the inequalities hold for all such k if the second inequality holds when $k = \alpha$. It follows that for $k \in [t, \alpha]$, Lemma 4 again applies to show that $f(x)$ has no factor of degree k.

We are left with considering the possibility of a linear factor (i.e, $k = 1$). We use Lemmas 2 and 3 for this purpose. The computation for $n \leq 2479$ (and beyond) was a direct application of these lemmas. We take two primes p_1 and p_2, p_1 being the largest prime factor of n and p_2 being the largest prime factor of $n+1$. We check if Lemma 2 applies with $k = 1$, $\ell = 0$, and $p = p_1$. If it does, we're done as $f(x)$ then cannot have a linear factor. Otherwise, we check if Lemma 3 applies with $k = 1$, $\ell = -1$, and $p = p_2$. In every case except $n = 2$ and $n = 3$, one of these two lemmas applied to show that $f(x)$ does not have a linear factor. For $n = 2$ and $n = 3$, one can apply Lemma 1 directly with $p = 3$ (in fact, $f(x)$ is Eisenstein with respect to 3 in these cases). We deduce that $f(x)$ cannot have a linear factor.

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