Notes for Seminar:

*The Odd Covering Problem and Its Relatives, Part II*

**Schinzel’s Theorem:** If there is an \( f(x) \in \mathbb{Z}[x] \) with \( f(1) \neq -1 \) such that \( f(x)x^n + 1 \) is reducible for all \( n \geq 0 \), then there is an odd covering of the integers.

**Notation:** Let \( \zeta_n = e^{2\pi i/n} \) and \( \Phi_n(x) = \prod_{1 \leq k \leq n, \gcd(k,n)=1} (x - \zeta_n^k) \).

**Lemma 1:** \( \Phi_n(x) = \prod_{d|n} (x^{d - 1})^{\mu(n/d)} = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)} \).

**Proof:** The factor \( x - \zeta_n^k \) is a factor of \( x^{n/d} - 1 \) precisely when \( n/d \) is a multiple of \( n/\gcd(n,k) \) (i.e., when \( d|\gcd(n,k) \)). Thus, \( x - \zeta_n^k \) appears in the right-most product above with exponent \( \sum_{d|\gcd(n,k)} \mu(d) \). The rest is clear.

**Lemma 2:** \( \Phi_{pm}(x) = \begin{cases} \Phi_n(x^p) & \text{if } p | n \\ \Phi_n(x^p)/\Phi_n(x) & \text{if } p \nmid n \end{cases} \).

**Proof:** Use Lemma 1. If \( p|n \), then \( \Phi_{pm}(x) = \prod_{pd|m} (x^{pd} - 1)^{\mu(pm/pd)} = \prod_{d|m} (x^{pd} - 1)^{\mu(n/d)} = \Phi_n(x^p) \). If \( p \nmid n \), then \( \Phi_{pm}(x) = \prod_{pd|m} (x^{pd} - 1)^{\mu(pm/pd)} \prod_{d|m} (x^d - 1)^{\mu(n/d)} = \Phi_n(x^p)/\Phi_n(x) \).

**Lemma 3:** Suppose \( m \) and \( n \) are integers with \( m/n = p^r \) for some prime \( p \) and some positive integer \( r \). Then \( \Phi_m(\zeta_n) = \alpha \Phi_n \) for some \( \alpha \in \mathbb{Z}[\zeta_n] \).

**Proof:** Consider three cases: (i) \( m = pn \) and \( p \nmid n \), (ii) \( m = p^r n \) with \( r > 1 \) and \( p \nmid n \), and (iii) \( m = p^r t \) and \( n = p^s t \) with \( u > v > 0 \). Let \( \xi \) denote an arbitrary primitive \( n \)th root of 1 (so \( \xi \in \mathbb{Z}[\zeta_n] \)). For (i), observe that Lemma 2 implies

\[
\Phi_m(x) = \frac{\Phi_n(x^p)}{\Phi_n(x)} = \prod_{1 \leq k \leq n, \gcd(k,n)=1} \left( \frac{x^p - \xi^{kp}}{x^p - \xi^k} \right) = \prod_{1 \leq k \leq n, \gcd(k,n)=1} (x^{p-1} + \xi^{k} x^{p-2} + \xi^{2k} x^{p-3} + \cdots + \xi^{(p-1)k})
\]

In particular, \( \Phi_m(\zeta_n) \) has the factor (take \( k = 1 \)) \( p^r \). For (ii), use Lemma 2 again to obtain \( \Phi_m(\zeta_n) = \Phi_n(\zeta_n^{p^{r-1}}) \) and apply the argument for (i) with \( \xi = \zeta_n^{p^{r-1}} \). For (iii), use Lemma 2 as before to obtain \( \Phi_m(\zeta_n) = \Phi_{p^{r-1}}(\zeta_n^{p^{r-1}}) = \Phi_{p^{r-1}}(\zeta) \). Now, cases (i) and (ii) imply \( \Phi_m(\zeta_n) = \alpha \Phi_n \) for some \( \alpha \in \mathbb{Z}[\zeta_n] \) (since \( \zeta_n = \zeta_n^{p^{r-1}} \)).

**Lemma 4:** Let \( p \) be a prime, and let \( m \) be a positive integer such that \( p \) divides \( m \). Then \( x^p = \zeta_m \) has no solutions \( x \in \mathbb{Q}(\zeta_m) \).

**Proof:** Let \( \zeta = \zeta_m \). The roots of \( x^p - \zeta = 0 \) are \( \zeta_m^{p^k} \) where \( 0 \leq k \leq p - 1 \). Note that \( \zeta_p = \zeta_m^{p^p} \subseteq \mathbb{Q}(\zeta) \). Thus, \( x^p = \zeta \) and \( x \in \mathbb{Q}(\zeta) \) imply \( \zeta_m \in \mathbb{Q}(\zeta) \), a contradiction.

**Lemma 5:** Suppose \( f(x) = -g(x)p \) for some prime \( p \) and \( f(x)x^n + 1 \) is divisible by \( \Phi_m(x) \) where \( p|m \). Then \( n \equiv 0 \pmod{p} \).

**Proof:** Assume \( p \nmid n \). Then there are integers \( u \) and \( v \) such that \( -nu + pv = 1 \). Since also \( f(\zeta)^n + 1 = 0 \), we deduce that \( -f(\zeta) = \zeta^{-n} \). Hence, \( (g(\zeta)^u \zeta^n)^p = \zeta^{-nu+pv} = \zeta \). Thus, \( x^p = \zeta \) has a solution \( x \in \mathbb{Q}(\zeta) \), contradicting Lemma 4.