On an Irreducibility Theorem
of A. Schinzel
Associated with
Coverings of the Integers

by

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Coverings of the Integers:

A covering of the integers is a system of congruences

\[ x \equiv a_j \pmod{m_j} \]

having the property that every integer satisfies at least one such congruence.

Example 1:

\[ x \equiv 0 \pmod{2} \]
\[ x \equiv 1 \pmod{2} \]
Example 2:

\[ x \equiv 0 \pmod{2} \]
\[ x \equiv 2 \pmod{3} \]
\[ x \equiv 1 \pmod{4} \]
\[ x \equiv 1 \pmod{6} \]
\[ x \equiv 3 \pmod{12} \]

Open Problem:

Does there exist an “odd covering” of the integers, a finite covering consisting of distinct odd moduli \( > 1 \)?

Erdős: $25 \ (\text{for proof none exists})$

Selfridge: $2000 \ (\text{for explicit example})$
Sierpinski’s Application:

There exist infinitely many (even a positive proportion of) positive integers \( k \) such that \( k \times 2^n + 1 \) is composite for all non-negative integers \( n \).

Selfridge’s Example: \( k = 78557 \)  
(smallest known)

Polynomial Question: Does there exist a polynomial \( f(x) \in \mathbb{Z}[x] \) such that \( f(x)x^n + 1 \) is reducible for all non-negative integers \( n \)?

Require: \( f(1) \neq -1 \)

Answer: Nobody knows.
Schinzel’s Example:

\[(5x^9 + 6x^8 + 3x^6 + 8x^5 + 9x^3 + 6x^2 + 8x + 3)x^n + 12\]

is reducible for all non-negative integers \(n\)

Schinzel’s Theorem: If there is an \(f(x) \in \mathbb{Z}[x]\) such that \(f(1) \neq -1\) and \(f(x)x^n + 1\) is reducible for all non-negative integers \(n\), then there is an odd covering of the integers.

Key Idea: Investigate non-cyclotomic factors of \(f(x)x^n + 1\), and show that typically the non-cyclotomic part of \(f(x)x^n + 1\) is irreducible.
**Key Idea:** Investigate non-cyclotomic factors of \( f(x)x^n + 1 \), and show that typically the non-cyclotomic part of \( f(x)x^n + 1 \) is irreducible.

**Observation:** One gets a non-trivial factorization of \( f(x)x^n + 1 \) when one of the following holds:

(i) \( f(x) \) is minus a \( p \)-th power and \( p \mid n \)

(ii) \( f(x) \) is 4 times a 4th power and \( 4 \mid n \).

**Note:** \( 4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1) \)

**Schinzel:** For fixed \( f(x) \in \mathbb{Z}[x] \) and \( n \) sufficiently large, the non-cyclotomic part of \( f(x)x^n + 1 \) is irreducible unless (i) or (ii) holds.
**Schinzel:** For fixed \( f(x) \in \mathbb{Z}[x] \) and \( n \) sufficiently large, the non-cyclotomic part of \( f(x)x^n + 1 \) is irreducible unless one of the following holds:

(i) \( f(x) \) is minus a \( p \)th power and \( p | n \)

(ii) \( f(x) \) is 4 times a 4th power and \( 4 | n \).

**Schinzel’s Example:**

\[
(5x^9 + 6x^8 + 3x^6 + 8x^5 + 9x^3 + 6x^2 + 8x + 3)x^n + 12
\]

is reducible for all non-negative integers \( n \)

In fact, for each \( n \), the above polynomial is divisible by one of

\[
\Phi_k(x) \quad \text{where} \quad k \in \{2, 3, 4, 6, 12\}.
\]
Notation:

irreducibility will be over the integers

\[ f(x) = \sum_{j=0}^{n} a_j x^j, \text{ then } \|f\| = \sqrt{\sum_{j=0}^{n} a_j^2} \]

\( \tilde{f}(x) = x^{\deg f} f(1/x) \)

\( \tilde{f}(x) \) will be called the reciprocal of \( f(x) \)

\( f(x) \) reciprocal means \( \tilde{f}(x) = \pm f(x) \)

the non-reciprocal part of \( f(x) \) is \( f(x) \) removed of its irreducible reciprocal factors (sort of)
\[ \tilde{f}(x) = x^{\deg f} f(1/x) \]

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**Comment:** Given \( f(x) \in \mathbb{Z}[x] \), if \( n \) is sufficiently large and \( f(x)x^n + 1 \) is divisible by an irreducible reciprocal polynomial \( g(x) \), then \( g(x) \) is cyclotomic.

Therefore, for \( n \) large, the non-cyclotomic part of \( f(x)x^n + 1 \) and non-reciprocal part of \( f(x)x^n + 1 \) are the same.
**Schinzel:** For fixed $f(x) \in \mathbb{Z}[x]$ and $n$ sufficiently large, the non-reciprocal part of $f(x)x^n + 1$ is irreducible unless one of the following holds:

(i) $f(x)$ is minus a $p$th power and $p|n$

(ii) $f(x)$ is 4 times a 4th power and $4|n$.

**Forget Everything Said Except Note:**

We want to say something about when the non-reciprocal part of $f(x)x^n + 1$ is irreducible.
**Theorem (F., Ford, Konyagin).** Let \( f(x) \) and \( g(x) \) be in \( \mathbb{Z}[x] \) with

\[
f(0) \neq 0, \ g(0) \neq 0, \text{ and } \gcd(f(x), g(x)) = 1.
\]

Let \( r_1 \) and \( r_2 \) denote the number of non-zero terms in \( f(x) \) and \( g(x) \), respectively. If

\[
n \geq \max \left\{ 2 \times 5^{2N-1}, 2 \max \{ \deg f, \deg g \} \left( 5^{N-1} + \frac{1}{4} \right) \right\}
\]

where

\[
N = 2 \| f \|^2 + 2 \| g \|^2 + 2r_1 + 2r_2 - 7,
\]

then the non-reciprocal part of \( f(x)x^n + g(x) \) is irreducible unless one of the following holds:

(i) The polynomial \(-f(x)g(x)\) is a \( p \)th power for some prime \( p \) dividing \( n \).

(ii) One of \( \pm f(x) \) or \( \pm g(x) \) is a 4th power, the other is 4 times a 4th power, and \( 4 \mid n \).
Capelli's Theorem: Let $F$ be a field. The polynomial $x^n + a \in F[x]$ is reducible if and only if either (i) $a$ is minus a $p$th power in $F$ for a prime $p$ dividing $n$ or (ii) $a$ is 4 times a 4th power in $F$ and 4 divides $n$.

Idea: Take $F = \mathbb{Q}(x)$. Instead of $f(x)x^n + g(x)$, consider $f(x)y^n + g(x)$ which is reducible in $\mathbb{Q}(x)$ if and only if $y^n + f(x)/g(x)$ is. Apply Capelli’s Theorem.

Problem: If $f(x)x^n + g(x)$ is reducible, then $f(x)y^n + g(x)$ may be irreducible

Want:

If the non-reciprocal part of $f(x)x^n + g(x)$ is reducible, then $f(x)y^n + g(x)$ is reducible.
Another Related Problem:

Suppose that $a_1, a_2, \ldots, a_r$ are distinct non-negative integers written in increasing order and that we wish to determine an integer $k \geq 2$ such that

$$a_j \mod k < k/2 \quad \text{for each } j \in \{1, 2, \ldots, r\}.$$

The value $k = 2a_r + 1$ satisfies this property.

Examples of sets $S = \{a_1, \ldots, a_r\}$ for which this choice of $k \geq 2$ is minimal are given by

$$\{3, 5\} \quad \text{and} \quad \{50, 68, 125\}.$$

Fix $r$. Is it true that if $a_r$ is sufficiently large, then one can always find a smaller $k$ with this property?
Want:
If the non-reciprocal part of \( f(x)x^n + g(x) \) is reducible, then \( f(x)y^n + g(x) \) is reducible.

Let \( F(x) = f(x)x^n + g(x) \). If the non-reciprocal part of \( F(x) \) is reducible, then there are non-reciprocal \( u(x) \) and \( v(x) \) with

\[
F(x) = u(x)v(x).
\]

Consider

\[
W(x) = u(x)\tilde{v}(x).
\]

Then

\[
F(x)\tilde{F}(x) = u(x)v(x)\tilde{u}(x)\tilde{v}(x) = W(x)\tilde{W}(x).
\]

Compare the coefficients of \( x^\text{deg}F \) on the left and right. On the left it is \( \|F\|^2 \), and on the right it is \( \|W\|^2 \). Hence,

\[
\|W\| = \|F\|.
\]
\[ F(x) = f(x)x^n + g(x) \]

\[ F(x) = u(x)v(x) \quad \text{and} \quad W(x) = u(x)\tilde{v}(x) \]

\[ \|W\| = \|F\| \]

Hence, the number of non-zero terms among both \( F(x) \) and \( W(x) \) is bounded by

\[ \|f\|^2 + \|g\|^2 + r_1 + r_2, \]

which is independent of \( n \).

Take a positive integer \( k \) (not too small and not too large) such that each exponent in \( F, W, \tilde{F}, \) and \( \tilde{W} \) is < \( k/2 \) when reduced modulo \( k \).
Exponents in $F$, $W$, $\tilde{F}$, $\tilde{W}$ mod $k$ are $< k/2$.

$$F(x) = \sum_{j=0}^{r} a_j x^{d_j} \rightarrow G_1(x, y) = \sum_{j=0}^{r} a_j x^{d_j} y^{\ell_j}$$

$$\tilde{F}(x) = \sum_{j=0}^{r} a_j x^{d_{r-d_j}} \rightarrow G_2(x, y) = \sum_{j=0}^{r} a_j x^{d_j} y^{\ell_j'}$$

$G_1(x, x^k) = F(x)$ and $G_2(x, x^k) = \tilde{F}(x)$

$$G_1(x, y)G_2(x, y) = \sum_{j=0}^{t} g_j(x)y^j$$

$\deg g_j(x) < k$ for all $j$

$$\sum_{j=0}^{t} g_j(x)x^{kj} = G_1(x, x^k)G_2(x, x^k) = F(x)\tilde{F}(x)$$
\[W(x) = \sum_{j=0}^{s} b_j x^{e_j} \rightarrow H_1(x, y) = \sum_{j=0}^{r} a_j x^{e_j} y^{m_j}\]

\[\widetilde{W}(x) = \sum_{j=0}^{s} b_j x^{e_{r-e_j}} \rightarrow H_2(x, y) = \sum_{j=0}^{s} b_j x^{e_j} y^{m_j'}\]

\[H_1(x, x^k) = W(x) \quad \text{and} \quad H_2(x, x^k) = \widetilde{W}(x)\]

\[H_1(x, y)H_2(x, y) = \sum_{j=0}^{t'} h_j(x)y^j\]

\[\deg h_j(x) < k \quad \text{for all } j\]

\[\sum_{j=0}^{t'} h_j(x)x^{k_j} = H_1(x, x^k)H_2(x, x^k) = W(x)\widetilde{W}(x)\]
\[ \sum_{j=0}^{t} g_j(x) x^{kj} = G_1(x, x^k) G_2(x, x^k) = F(x) \tilde{F}(x) \]

\[ \sum_{j=0}^{t'} h_j(x) x^{kj} = H_1(x, x^k) H_2(x, x^k) = W(x) \tilde{W}(x) \]

\[ \sum_{j=0}^{t} g_j(x) x^{kj} = \sum_{j=0}^{t'} h_j(x) x^{kj} \]

\[ g_j(x) = h_j(x) \quad \text{for all } j \]

\[ G_1(x, y) G_2(x, y) = \sum_{j=0}^{t} g_j(x) y^j \]

\[ H_1(x, y) H_2(x, y) = \sum_{j=0}^{t'} h_j(x) y^j \]

\[ G_1(x, y) G_2(x, y) = H_1(x, y) H_2(x, y) \]
\[ G_1(x, y)G_2(x, y) = H_1(x, y)H_2(x, y) \]

\[ G_1(x, x^k) = F(x) \quad \& \quad G_2(x, x^k) = \tilde{F}(x) \]
\[ H_1(x, x^k) = W(x) \quad \& \quad H_2(x, x^k) = \tilde{W}(x) \]

\[ G_1, G_2, H_1, \& H_2 \] are pairwise distinct.

Each is reducible.

\[ F(x) = \sum_{j=0}^{r} a_j x^{d_j}, \quad G_1(x, y) = \sum_{j=0}^{r} a_j x^{\bar{d}_j} y^{\ell_j} \]

\[ F(x) = f(x)x^n + g(x) \]

\[ G_1(x, y) = f(x)x^d y^\ell + g(x) \]
\[ F(x) = f(x)x^n + g(x) \]

\[ G_1(x, y) = f(x)x^d y^\ell + g(x) \]

**Conclusion:** If the non-reciprocal part of \( f(x)x^n + g(x) \) is reducible, then

\[ f(x)x^d y^\ell + g(x) \]

is reducible.

Apply Capelli’s Theorem.