Seminar Notes: On Nicol’s sequence of reducible polynomials

**Problem:** Does this ever end?

\[ 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} + \cdots \]

**Goal:** Justify the answer (whatever it is).

**Definitions and Notation:** Given \( f(x) \in \mathbb{C}[x] \) with \( f \neq 0 \), \( \tilde{f}(x) = x^{\deg f} f(1/x) \) is the reciprocal of \( f(x) \). If \( f = \pm \tilde{f} \), then \( f \) is called reciprocal.

**Comment:** If \( f \) is reciprocal and \( \alpha \) is a root of \( f \), then \( 1/\alpha \) is a root of \( f \).

**Two-Step Approach:**

1. Handle reciprocal factors (there are none).
2. Handle non-reciprocal factors (there is no more than one).

**Step 1:** Take \( g(x) = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} \).
   - If \( f \) is an irreducible reciprocal factor of \( F(x) = x^n + g(x) \), then it divides \( \tilde{F}(x) \).
   - So it divides \( g(x)\tilde{g}(x) - x^{\deg g} \).
   - So it is either \( x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \) or \( x^{64} + x^{61} - x^{60} + x^{54} - \cdots - x^{43} + 2x^{42} + x^{41} - \cdots + x^{10} - x^4 + x^3 + 1 \).
   - In the first case, check \( 0 \leq n \leq 6 \). Done.
   - In the second case, \( f \) has a root \( \alpha = 0.58124854 - 0.96349774i \) with \( 1.25 < |\alpha| < 1.126 \). Observe that \( |g(\alpha)| < g(1.126) < 231 < 1.125^{17} < |\alpha|^{47} \). So \( F(\alpha) \neq 0 \) for all \( n \geq 1 \).

**Step 2:** Assume \( F(x) = x^n + g(x) \) is reducible. Let \( a(x) \) be an irreducible non-reciprocal factor. If \( \tilde{a}(x) \) divides \( F \), write \( F(x) = u(x)v(x) \) where \( \tilde{a}(x) \nmid u(x) \) and \( a(x) \nmid v(x) \). If \( \tilde{a}(x) \) does not divide \( F \), consider an irreducible non-reciprocal \( b(x) \) such that \( a(x)b(x) \) divides \( F \). If \( \tilde{b}(x) \) divides \( F \), write \( F(x) = u(x)v(x) \) where \( \tilde{b}(x) \nmid u(x) \) and \( b(x) \nmid v(x) \). If \( \tilde{a}(x) \) and \( \tilde{b}(x) \) do not divide \( F \), write \( F(x) = u(x)v(x) \) where \( a(x)|u(x) \) and \( b(x)|v(x) \). In all cases, we may take both \( u \) and \( v \) to have a positive leading coefficient.
   - Can \( F \) have a reciprocal factor? Maybe, but \( u \) and \( v \) are non-reciprocal.
   - **Lemma.** The polynomial \( w(x) = u(x)\tilde{v}(x) \) has the following properties:
     (i) \( w \neq \pm F \) and \( w \neq \pm \tilde{F} \).
     (ii) \( w\tilde{w} = F\tilde{F} \).
     (iii) \( w(1) = \pm F(1) \).
     (iv) \( ||w|| = ||F|| \).
     (v) \( w \) is a 0, 1-polynomial with the same number of non-zero terms as \( F \).
Proof of (v). If \( F(x) = \sum_{j=1}^{r} a_j x^{d_j} \) and \( w(x) = \sum_{j=1}^{s} b_j x^{e_j} \), then

\[
\left( \sum_{j=1}^{s} b_j \right)^2 \leq \left( \sum_{j=1}^{s} b_j^2 \right)^2 = \left( \sum_{j=1}^{s} a_j^2 \right)^2 = \left( \sum_{j=1}^{s} a_j \right)^2 = \left( \sum_{j=1}^{s} b_j \right)^2.
\]

\[\blacksquare\]

- If \( n \geq 83 \), then \( \tilde{F} = 1 + x^3 + x^{15} + x^{16} + x^{32} + x^{33} + x^{34} + x^{35} + x^m + \cdots \) where \( m \geq 48 \).

- What can \( w \) and \( \tilde{w} \) be given (v), (ii), and \( n \geq 83 \)?

\[
\begin{align*}
w(x) &= 1 + x^3 + \cdots + x^n \\
\tilde{w}(x) &= 1 + \cdots + x^{n-3} + x^n \\
w(x) &= 1 + x^3 + x^{15} + \cdots + x^n \\
\tilde{w}(x) &= 1 + \cdots + x^{n-15} + x^{n-3} + x^n \\
w(x) &= 1 + x^3 + x^{15} + x^{16} + \cdots + x^n \\
\tilde{w}(x) &= 1 + \cdots + x^{n-16} + x^{n-15} + x^{n-3} + x^n \\
&\vdots
\end{align*}
\]

- Given (i), “the non-reciprocal part is irreducible”.

**Comment:** In general, consider a 0, 1-polynomial \( g(x) \) with the property that \( g(x) \) is irreducible over the set of 0, 1-polynomials (that is, \( g(x) \) is not the product of two 0, 1-polynomials of degree > 0). Then the non-reciprocal part of \( F(x) = x^n + g(x) \) is irreducible if \( n > 3 \deg g \).