APPLICATIONS OF PADÉ APPROXIMATIONS OF \((1 - z)^k\) TO NUMBER THEORY

by Michael Filaseta

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General Areas of Applications:
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- irrationality measures
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- diophantine equations
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- $k$-free numbers in short intervals
- $k$-free values of polynomials and binary forms
- the $abc$-conjecture
What are the Padé approximations of \((1 - z)^k\)?
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**Answer:** Rational functions that give good approximations to \((1 - z)^k\) near the origin.
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P - (1 - z)^k Q = z^m E
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**Important Equation:**

\[ P_r - (1 - z)^k Q_r = z^{2r+1} E_r \]

\[ \text{deg } P_r = \text{deg } Q_r = r < k, \quad \text{deg } E_r = k - r - 1 \]
Some Properties of the Polynomials:

(i) \( P_r(z), (-z)^k Q_r(z), \) and \( z^{2r+1} E_r(z) \) satisfy

\[
z(z-1)y'' + (2r(1-z) - (k-1)z)y' + r(k+r)y = 0.
\]

(ii) \( Q_r(z) = \sum_{j=0}^{r} \binom{2r-j}{r} \binom{k-r+j-1}{j} z^j \)

(iii) \( Q_r(x) = \frac{(k+r)!}{(k-r-1)! r! r!} \int_{0}^{1} (1-t)^r t^{k-r-1} (1-t+xt)^r \, dt \)

(iv) \( P_r(x) Q_{r+1}(x) - Q_r(x) P_{r+1}(x) = cx^{2r+1} \)
\[ P_r - (1 - z)^k Q_r = z^{2r+1} E_r \]
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\]

**Warning:** In the applications you are about to see, this identity is used to get a result of the type wanted. Typically, a closer analysis of these polynomials or even a variant of the polynomials is needed to obtain the currently best known results in these applications.
Irrationality measures:
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Theorem (Liouville): Fix $\alpha \in \mathbb{R} - \mathbb{Q}$ with $\alpha$ algebraic and of degree $n$. Then there is a constant $C = C(\alpha) > 0$ such that

$$\left| \alpha - \frac{a}{b} \right| > \frac{C}{b^n}$$

where $a$ and $b$ with $b > 0$ are arbitrary integers.
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Irrationality measures:

**Theorem (Roth):** Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with $\alpha$ algebraic. Then there is a constant $C = C(\alpha, \varepsilon) > 0$ such that

$$\left| \alpha - \frac{a}{b} \right| > \frac{C}{b^{2+\varepsilon}}$$

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Irrationality measures:

**Theorem (Roth):** Fix $\varepsilon > 0$ and $\alpha \in \mathbb{R} - \mathbb{Q}$ with $\alpha$ algebraic. Then there is a constant $C = C' (\alpha, \varepsilon) > 0$ such that

$$\left| \alpha - \frac{a}{b} \right| > \frac{C}{b^{2 + \varepsilon}}$$

where $a$ and $b$ with $b > 0$ are arbitrary integers.

**Comment:** Liouville’s result is effective; Roth’s is not.
Irrationality measures:

Theorem (Baker): For $a$ and $b$ integers with $b > 0$,

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{C}{b^{2.955}}$$

where $C = 10^{-6}$. 
Irrationality measures:

Theorem (Baker): For \( a \) and \( b \) integers with \( b > 0 \),

\[
\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{1}{10^6 b^{2.955}}.
\]
Irrationality measures:

**Theorem (Chudnovsky):** For $a$ and $b$ integers with $b > 0$,

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{1}{c \cdot b^{2.43}}.$$
Irrationality measures:

**Theorem (Bennett):** For $a$ and $b$ integers with $b > 0$,

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{1}{c \cdot b^{2.47}}.$$
Irrationality measures:

Theorem (Bennett): For $a$ and $b$ integers with $b > 0$,

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\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{1}{4 \cdot b^{2.47}}.
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**Comment:** Similar explicit estimates have also been made for certain other cube roots.
The Basic Approach:
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\[ P_r - (1 - z)^k Q_r = z^{2r+1} E_r \]
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\[ P_r - (1 - z)^{1/3}Q_r = z^{2r+1}E_r \]
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\[ \uparrow \]

\[ 3/128 \]
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Rearrange and Normalize to Integers
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\[ \sqrt[3]{2} b_r - a_r = \text{small} \]
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Wait!!
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Wait!! I thought we wanted that LARGE!!
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What's small\(_r\)? Let \( b \) be a positive integer.
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What’s \text{small}_r? Let \(b\) be a positive integer. By choosing \(r\) right, one can obtain

$$\text{small}_r < \frac{1}{2b b_r} \quad \text{and} \quad b_r < cb^{1.47}.$$
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$$\left| \frac{3\sqrt{2} - a}{b} \right| \geq \left| \frac{a_r}{b_r} - \frac{a}{b} \right| - \left| 3\sqrt{2} - \frac{a_r}{b_r} \right| > \frac{1}{2b b_r}$$
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Theorem (Bennett): For $a$ and $b$ integers with $b > 0$,

$$\left| \frac{3\sqrt{2} - \frac{a}{b}}{b} \right| > \frac{1}{4 \cdot b^{2.47}}.$$
Diophantine equations:

Theorem (Bennett): For $a$ and $b$ integers with $b \neq 0$,

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Diophantine equations:

Theorem (Bennett): For $a$ and $b$ integers with $b \neq 0$,

$$\left| \sqrt[3]{2} - \frac{a}{b} \right| > \frac{1}{4 \cdot |b|^{2.5}}.$$
Diophantine equations:

**Theorem (Bennett):** For $a$ and $b$ integers with $b \neq 0$,

$$\left|\frac{\sqrt[3]{2} - \frac{a}{b}}{b}\right| > \frac{1}{4 \cdot |b|^{2.5}}.$$

$$x^3 - 2y^3 = n$$
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$$x^3 - 2y^3 = n, \quad y \neq 0$$

$$\left| \frac{3\sqrt{2} - \frac{x}{y}}{\sqrt{2}} \right| \left| \frac{3\sqrt{2}e^{2\pi i/3} - \frac{x}{y}}{\sqrt{2}e^{2\pi i/3}} \right| \left| \frac{3\sqrt{2}e^{4\pi i/3} - \frac{x}{y}}{\sqrt{2}e^{4\pi i/3}} \right| = \frac{|n|}{|y|^3}$$
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$$\left| \frac{3\sqrt{2} - x}{y} \right| \left| \frac{3\sqrt{2}e^{2\pi i/3} - x}{y} \right| \left| \frac{3\sqrt{2}e^{4\pi i/3} - x}{y} \right| = \frac{|n|}{|y|^3}$$

$$\left| \frac{3\sqrt{2} - x}{y} \right| < \frac{|n|}{|y|^3}$$
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\[
\left| \frac{3\sqrt{2}}{y} - \frac{x}{y} \right| \left| \frac{3\sqrt{2}e^{2\pi i/3}}{y} - \frac{x}{y} \right| \left| \frac{3\sqrt{2}e^{4\pi i/3}}{y} - \frac{x}{y} \right| = \frac{|n|}{|y|^3}.
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\[ x^3 - 2y^3 = n, \quad y \neq 0 \]

\[ \left| \frac{\sqrt[3]{2}}{y} \right| = \frac{|n|}{|y|^3} \]

\[ \frac{1}{4|y|^{2.5}} < \left| \frac{\sqrt[3]{2} - \frac{x}{y}}{y} \right| < \frac{|n|}{|y|^3} \]

\[ |y|^{1/2} < 4|n| \]
Diophantine equations:

\[ x^3 - 2y^3 = n, \quad y \neq 0 \]

\[
\left| 3\sqrt{2} - \frac{x}{y} \right| \leq \left| 3\sqrt{2}e^{2\pi i/3} - \frac{x}{y} \right| \leq \left| 3\sqrt{2}e^{4\pi i/3} - \frac{x}{y} \right| = \frac{|n|}{|y|^3}
\]

\[
\frac{1}{4|y|^{2.5}} < \left| 3\sqrt{2} - \frac{x}{y} \right| < \frac{|n|}{|y|^3}
\]

\[
|y|^{1/2} < 4|n| \implies |y| < 16n^2
\]
Diophantine equations:

\[ x^3 - 2y^3 = n, \quad y \neq 0 \]

\[
\left| \frac{3\sqrt{2}}{y} - \frac{x}{y} \right| < \left| \frac{3\sqrt{2}e^{2\pi i/3}}{y} - \frac{x}{y} \right| < \left| \frac{3\sqrt{2}e^{4\pi i/3}}{y} - \frac{x}{y} \right| = \frac{|n|}{|y|^3}
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|y|^{1/2} < 4|n| \implies |y| < 16n^2
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Diophantine equations:

**Theorem:** Let $n$ be a non-zero integer. If $x$ and $y$ are integers satisfying $x^3 - 2y^3 = n$, then $|y| < 16n^2$. 

Diophantine equations:

**Theorem (Bennett):** If $a$, $b$, and $n$ are integers with $ab \neq 0$ and $n \geq 3$, then the equation

$$|ax^n + by^n| = 1$$

has at most one solution in positive integers $x$ and $y$. 
Waring’s Problem:
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Theorem (Beukers): If \( k > 4 \), then

\[ \| \left( \frac{3}{2} \right)^k \| > 0.5358^k. \]
Waring’s Problem:

**Known:**

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**Theorem (Dubitskas):** If $k > 4$, then

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The factorization of $n(n + 1)$:
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**Well-Known:** The largest prime factor of $n(n + 1)$ tends to infinity with $n$. 
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**Well-Known:** The largest prime factor of $n(n + 1)$ tends to infinity with $n$.

Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N$ such that if $n \geq N$ and

$$n(n + 1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m$$

for some integer $m$, then $m > 1$. 
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**Lehmer:** Gave some explicit estimates:
Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N$ such that if $n \geq N$ and

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\[ n(n + 1) \] divisible only by primes $\leq 11 \implies n \leq \]
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**Lehmer:** Gave some explicit estimates:

$n(n + 1)$ divisible only by primes $\leq 11 \implies n \leq 9800$

... only by primes $\leq 41 \implies n \leq \ldots$
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**Lehmer:** Gave some explicit estimates:

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Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

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\[\text{abc-conjecture} \implies \theta = \]
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(ineffective)
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Effective Approach:
Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and

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Effective Approach: (Linear Forms of Logarithms)
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**Effective Approach:** (Linear Forms of Logarithms)

$$\theta = \frac{c}{\log \log n}$$
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Effective Approach: (Linear Forms of Logarithms)

$$\theta = \frac{c}{\log \log n}$$

Problem: Can we narrow the gap between these ineffective and effective results?
Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and 

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Theorem (Bennett, F., Trifonov): If \( n \geq 9 \) and
\[
n(n + 1) = 2^k 3^\ell m,
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then
\[
m \geq \]
Want: Let $p_1, p_2, \ldots, p_r$ be primes. There is an $N = N(\theta, p_1, \ldots, p_r)$ such that if $n \geq N$ and
\[ n(n + 1) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} m \]
for some integer $m$, then $m > n^\theta$.

Theorem (Bennett, F., Trifonov): If $n \geq 9$ and
\[ n(n + 1) = 2^k 3^\ell m, \]
then
\[ m \geq n^{1/4}. \]
Conjecture: For \( n > 512 \),

\[
n(n + 1) = 2^u 3^v m \implies m > \sqrt{n}.
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Comment: The conjecture has been verified for
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Conjecture: For $n > 512$,

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Comment: The conjecture has been verified for $512 < n \leq 10^{1000}$. 
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Main Idea: Find “small” integers \( P, Q, \) and \( E \) such that

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Obtain an upper bound on $3^k$. 
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Obtain an upper bound on $3^k$. Since $3^k m_1 \geq n$, it follows that $m_1$ and, hence, $m = m_1 m_2$ are not small.
The “small” integers $P$, $Q$, and $E$ are obtained through the use of Padé approximations for $(1 - x)^k$. 
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More precisely, one takes $z = 1/9$ in the equation

$$P_r(x) - (1 - x)^k Q_r(x) = x^{2r+1} E_r(x).$$
What’s Needed for the Method to Work:
What’s Needed for the Method to Work:

One largely needs to be dealing with two primes (like 2 and 3) with a difference of powers of these primes being small (like $3^2 - 2^3 = 1$).
Galois groups associated with classical polynomials:
There is polynomial \( f(x) \in \mathbb{Z}[x] \) such that the Galois group associated with \( f(x) \) is the symmetric group \( S_n \).

- D. Hilbert (1892) used his now classical Hilbert’s Irreducibility Theorem to show that for each integer \( n \geq 1 \), there is polynomial \( f(x) \in \mathbb{Z}[x] \) such that the Galois group associated with \( f(x) \) is the symmetric group \( S_n \).
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- Hilbert’s work and work of E. Noether (1918) began what has come to be known as Inverse Galois Theory.

- Van der Waerden showed that for “almost all” polynomials \( f(x) \in \mathbb{Z}[x] \), the Galois group associated with \( f(x) \) is the symmetric group \( S_n \).
Galois groups associated with classical polynomials:

- Schur showed $L_n^{(0)}(x)$ has Galois group $S_n$. 
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- Schur showed $L_n^{(0)}(x)$ has Galois group $S_n$.
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- Schur showed $\sum_{j=0}^{n} \frac{x^j}{j!}$ has Galois group $A_n$ if $4|n$.

- Schur did not find an explicit sequence of polynomials having Galois group $A_n$ with $n \equiv 2 \pmod{4}$.
Galois groups associated with classical polynomials:

Theorem (R. Gow, 1989): If $n \geq 2$ is even and

$$L_n^{(n)}(x) = \sum_{j=0}^{n} \binom{2n}{n-j} \frac{(-x)^j}{j!}$$

is irreducible, then the Galois group of $L_n^{(n)}(x)$ is $A_n$. 
Galois groups associated with classical polynomials:

**Theorem (R. Gow, 1989):** If \( n > 2 \) is even and

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\]

is irreducible, then the Galois group of \( L_n^{(n)}(x) \) is \( A_n \).

**Theorem (joint work with R. Williams):** For almost all positive integers \( n \) the polynomial \( L_n^{(n)}(x) \) is irreducible (and, hence, has Galois group \( A_n \) for almost all even \( n \)).
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**Comment:** The method had an ineffective component to it. We could show that if \( n \) is sufficiently large and \( L_n^{(n)}(x) \) is reducible, then \( L_n^{(n)}(x) \) has a linear factor. But we didn’t know what sufficiently large was.
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**Work in Progress with Trifonov:** There is an effective bound $N$ such that if $n \geq N$ and $n \equiv 2 \pmod{4}$, then $L_n^{(n)}(x)$ is irreducible.
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The Ramanujan-Nagell equation:
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Classical Ramanujan-Nagell Theorem: If \( x \) and \( n \) are integers satisfying

\[ x^2 + 7 = 2^n, \]

then

\[ x \in \{1, 3, 5, 11, 181\}. \]
The Ramanujan-Nagell equation:

Some Background: Beukers used a method “similar” to the approach for finding irrationality measures to show that $\sqrt{2}$ cannot be approximated too well by rationals $a/b$ with $b$ a power of $2$. This implies bounds for solutions to the Diophantine equation $x^2 + D = 2^n$ with $D$ fixed. This led to him showing that if $D \neq 7$, then the equation has at most 4 solutions. Related independent work by Apéry, Beukers, and Bennett establishes that for odd primes $p$ not dividing $D$, the equation $x^2 + D = p^n$ has at most 3 solutions. All of these are in some sense best possible (though more can and has been said).
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Problem: If $x^2 + 7 = 2^n m$ and $x$ is not in the set above, then can we say that $m$ must be large?
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Connection with \( n(n + 1) \) problem:
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\[
x^2 + 7 = 2^n m
\]

\[
\left( \frac{x + \sqrt{-7}}{2} \right) \left( \frac{x - \sqrt{-7}}{2} \right) = \left( \frac{1 + \sqrt{-7}}{2} \right)^{n-2} \left( \frac{1 - \sqrt{-7}}{2} \right)^{n-2} m
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↑

linear

↑

linear
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↑
linear
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prime
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Theorem (Bennett, F., Trifonov): If $x$, $n$ and $m$ are positive integers satisfying

$$x^2 + 7 = 2^n m \quad \text{and} \quad x \notin \{1, 3, 5, 11, 181\},$$

then

$$m \geq ???$$
Theorem (Bennett, F., Trifonov): If $x$, $n$ and $m$ are positive integers satisfying

$$x^2 + 7 = 2^nm \quad \text{and} \quad x \not\in \{1, 3, 5, 11, 181\},$$

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$$m \geq x^{1/2}.$$
Theorem (Bennett, F., Trifonov): If $x$, $n$ and $m$ are positive integers satisfying

$$x^2 + 7 = 2^n m \quad \text{and} \quad x \not\in \{1, 3, 5, 11, 181\},$$

then

$$m \geq x^{1/2}.$$ 

Comment: In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the $13^{\text{th}}$ power has absolute value $\approx 2.65$ and the powers themselves have absolute value $\approx 90.51$. 
$k$-free numbers in short intervals:
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**Problem:** Find $\theta = \theta(k)$ as small as possible such that, for $x$ sufficiently large, the interval $(x, x + x^\theta]$ contains a $k$-free number.
**k-free numbers in short intervals:**

**Problem:** Find $\theta = \theta(k)$ as small as possible such that, for $x$ sufficiently large, the interval $(x, x + x^\theta]$ contains a $k$-free number.

**Main Idea:** Show that there are integers in $(x, x + x^\theta]$ not divisible by the $k^{\text{th}}$ power of a prime. Consider primes in different size ranges. Deal with small primes and large primes separately.
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**Small Primes:** $p \leq z$
Problem: Find $\theta = \theta(k)$ as small as possible such that, for $x$ sufficiently large, the interval $(x, x + x^\theta]$ contains a $k$-free number.

Small Primes: $p \leq z$ where $z = x^\theta \sqrt{\log x}$
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The number of integers $n \in (x, x + x^\theta]$ divisible by such a $p^k$ is bounded by
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$$
\sum_{p \leq z} \left( \frac{x^\theta}{p^k} + 1 \right) \leq \left( \sum_{p \text{ prime}} \frac{x^\theta}{p^2} \right) + \pi(z)
$$
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$$\leq \left( \frac{\pi^2}{6} - 1 \right) x^\theta$$
Problem: Find \( \theta = \theta(k) \) as small as possible such that, for \( x \) sufficiently large, the interval \((x, x + x^\theta]\) contains a \( k \)-free number.

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The number of integers \( n \in (x, x + x^\theta] \) divisible by such a \( p^k \) is bounded by

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\sum_{p \leq z} \left( \frac{x^\theta}{p^k} + 1 \right) \leq \left( \sum_{p \text{ prime}} \frac{x^\theta}{p^2} \right) + \pi(z) \\
\leq \left( \frac{\pi^2}{6} - 1 \right) x^\theta < \frac{2}{3} x^\theta.
\]
Large Primes: $p \in (N, 2N]$, $N \geq z = x^\theta \sqrt{\log x}$
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\[
x < p^k m \leq x + x^\theta
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Large Primes: $p \in (N, 2N]$, $N \geq z = x^\theta \sqrt{\log x}$

$$x < p^k m \leq x + x^\theta \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^\theta}{p^k}$$
Large Primes: \( p \in (N, 2N], \; N \geq z = x^\theta \sqrt{\log x} \)

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where \( \|t\| = \min\{|t - \ell| : \ell \in \mathbb{Z}\} \)
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**Idea:** Show that there are few primes \( p \in (N, 2N] \) with \( x/p^k \) that close to an integer.
Large Primes: \( p \in (N, 2N], \ N \geq z = x^\theta \sqrt{\log x} \)

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x < p^km \leq x + x^\theta \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^\theta}{p^k}
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**Idea:** Show that there are few integers \( p \in (N, 2N] \) with \( x/p^k \) that close to an integer.
Large Primes: \( p \in (N, 2N], \; N \geq z = x^\theta \sqrt{\log x} \)

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x < p^k m \leq x + x^\theta \implies \frac{x}{p^k} < m \leq \frac{x}{p^k} + \frac{x^\theta}{p^k}
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where \( \| t \| = \min \{|t - \ell| : \ell \in \mathbb{Z}\} \)

**Idea:** Show that there are few integers \( u \in (N, 2N] \) with \( x/u^k \) that close to an integer.
\[ \frac{x}{u^k} < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]
\[\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}\]

Exponential Sums:
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

**Exponential Sums:** Let \( \delta \in (0, 1/2) \). Let \( f : \mathbb{R} \to \mathbb{R} \) be any function. Let \( S \) be a set of positive integers. Then for any positive integer \( J \leq 1/(4\delta) \), we get

\[
|\{u \in S : \|f(u)\| < \delta\}| \\
\leq \frac{\pi^2}{2(J + 1)} \sum_{1 \leq j \leq J} \left| \sum_{u \in S} e^{2\pi ij f(u)} \right| \\
+ \frac{\pi^2}{4(J + 1)}|S|.
\]
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

Differences:
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

Differences:

\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad \left\| \frac{x}{(u + a)^k} \right\| < \frac{x^\theta}{N^k} \]
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\[ \frac{x}{u^k} - \frac{x}{(u + a)^k} \]
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\[ \frac{x}{u^k} - \frac{x}{(u + a)^k} \sim \frac{ax}{u^{k+1}} \]
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

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\[ \frac{x}{u^k} - \frac{x}{(u + a)^k} \simeq \frac{ax}{u^{k+1}} \simeq \frac{ax}{N^{k+1}} \]
$$\left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x}$$

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\frac{x}{u^k} - \frac{x}{(u + a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{ax}{N^{k+1}}
\]

consider \( N = x^{1/k} \)
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

Differences:
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad \left\| \frac{x}{(u + a)^k} \right\| < \frac{x^\theta}{N^k} \]
\[ \frac{x}{u^k} - \frac{x}{(u + a)^k} \approx \frac{ax}{u^{k+1}} \approx \frac{a}{x^{1/k}} \]

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\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

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\[ \frac{x}{u^k} - \frac{x}{(u + a)^k} \asymp \frac{ax}{u^{k+1}} \asymp \frac{a}{x^{1/k}} \]

consider \( N = x^{1/k}, \ a < x^{1/(2k)} \)
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

Differences:

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\frac{x}{u^k} - \frac{x}{(u + a)^k} \approx \frac{ax}{u^{k+1}} \approx \frac{a}{x^{1/k}}
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consider \( N = x^{1/k}, \ a < x^{1/(2k)}, \ \theta \approx 1/k \)
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\frac{x}{u^k} - \frac{x}{(u + a)^k} \approx \frac{ax}{u^{k+1}} \approx \frac{a}{x^{1/k}}
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consider \( N = x^{1/k}, \ a < x^{1/(2k)}, \ \theta \approx 1/k \)

LHS small compared to RHS
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

“Modified” Differences:
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k} , \quad u \in (N, 2N] , \quad N \geq x^\theta \sqrt{\log x} \]

“Modified” Differences:

\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k} , \quad \left\| \frac{x}{(u + a)^k} \right\| < \frac{x^\theta}{N^k} \]
\[ \| \frac{x}{u^k} \| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

“Modified” Differences:

\[ \| \frac{x}{u^k} \| < \frac{x^\theta}{N^k}, \quad \| \frac{x}{(u + a)^k} \| < \frac{x^\theta}{N^k} \]

\[ \frac{x}{u^k} P - \frac{x}{(u + a)^k} Q \quad \text{small} \]
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

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\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad \left\| \frac{x}{(u+a)^k} \right\| < \frac{x^\theta}{N^k} \]

\[ \frac{x}{u^k} P - \frac{x}{(u+a)^k} Q \quad \text{small (but not too small)} \]
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“Modified” Differences:
\[ \frac{x}{u^k} < \frac{x^\theta}{N^k}, \quad \frac{x}{(u + a)^k} < \frac{x^\theta}{N^k} \]
\[ \frac{x}{u^k} P - \frac{x}{(u + a)^k} Q \quad \text{small (but not too small)} \]
\[ (u + a)^k P - u^k Q \quad \text{small (but not too small)} \]
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"Modified" Differences:

\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad \left\| \frac{x}{(u + a)^k} \right\| < \frac{x^\theta}{N^k} \]

\[ \frac{x}{u^k} P - \frac{x}{(u + a)^k} Q \quad \text{small (but not too small)} \]

\[ (u + a)^k P - u^k Q \quad \text{small (but not too small)} \]

consider \( P_r(z) - (1 - z)^k Q_r(z) \) with \( z = \frac{a}{u + a} \)
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

“Modified” Differences:

Theorem (Halberstam & Roth):
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

“Modified” Differences:

Theorem (Halberstam & Roth & Nair):
Theorem (Halberstam & Roth & Nair):
For $x$ sufficiently large, there is a $k$-free number in the interval $(x, x + x^{1/(2k)}].$
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

Modified Differences plus Divided Differences:
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

**Modified Differences plus Divided Differences:**

**Theorem (F. & Trifonov):** For \( x \) sufficiently large, there is a squarefree number in \( (x, x + cx^{1/5} \log x] \).
\[ \left\| \frac{x}{u^k} \right\| < \frac{x^\theta}{N^k}, \quad u \in (N, 2N], \quad N \geq x^\theta \sqrt{\log x} \]

**Modified Differences plus Divided Differences:**

**Theorem (F. & Trifonov):** For \( x \) sufficiently large, there is a squarefree number in \((x, x + cx^{1/5} \log x]\).

**Theorem (Trifonov):** For \( x \) sufficiently large, there is a \( k \)-free number in \((x, x + cx^{1/(2k+1)} \log x]\).
More General Theorem (F. & Trifonov): Let $k$ be an integer $\geq 2$, and let
\[ s \in \mathbb{Q} - \{-(k-1), -(k-2), \ldots, k-2, k-1\}. \]
Let $f(u) = X/u^s$. Suppose that
\[ N^s \leq X \quad \text{and} \quad \delta \leq cN^{-(k-1)}, \]
where $c > 0$ is small. Set
\[ S = \{ u \in \mathbb{Z} \cap (N, 2N] : \| f(u) \| < \delta \}. \]
Then
\[ |S| \ll_{k, s} X^{1/(2k+1)} N^{(k-s)/(2k+1)} \]
\[ + \delta X^{1/(6k+3)} N^{(6k^2+2k-s-1)/(6k+3)}. \]
More General Theorem (F. & Trifonov): Let $k$ be an integer $\geq 2$, and let

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Then

$$|S| \ll_{k,s} X^{1/(2k+1)} N^{(k-s)/(2k+1)}$$

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More General Theorem (F. & Trifonov): Let \( k \) be an integer \( \geq 2 \), and let

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N^s \leq X \quad \text{and} \quad \delta \leq cN^{-(k-1)},
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where \( c > 0 \) is small. Set

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S = \{u \in \mathbb{Z} \cap (N, 2N] : \|f(u)\| < \delta\}.
\]

Then

\[
|S| \ll_{k,s} X^{1/(2k+1)} N^{(k-s)/(2k+1)} + \delta X^{1/(6k+3)} N^{(6k^2+2k-s-1)/(6k+3)}.
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More General Theorem (F. & Trifonov): Let $k$ be an integer $\geq 2$, and let

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$N^s \leq X$ and $\delta \leq cN^{-(k-1)}$,

where $c > 0$ is small. Set

$S = \{u \in \mathbb{Z} \cap (N, 2N] : \|f(u)\| < \delta\}$.

Then

$|S| \ll_{k,s} X^{1/(2k+1)} N^{(k-s)/(2k+1)} + \delta X^{1/(6k+3)} N^{(6k^2+2k-s-1)/(6k+3)}$. 
$k$-free values of polynomials and binary forms:
\textit{k}-free values of polynomials and binary forms:

The method for obtaining results about gaps between \textit{k}-free numbers generalizes to \textit{k}-free values of polynomials.
$k$-free values of polynomials and binary forms:

The method for obtaining results about gaps between $k$-free numbers generalizes to $k$-free values of polynomials. Suppose $f(x) \in \mathbb{Z}[x]$ is irreducible and $\deg f = n$. 
The method for obtaining results about gaps between $k$-free numbers generalizes to $k$-free values of polynomials. Suppose $f(x) \in \mathbb{Z}[x]$ is irreducible and $\deg f = n$. In what follows, we suppose further that $f$ has no fixed $k^{\text{th}}$ power divisors.
The method for obtaining results about gaps between \( k \)-free numbers generalizes to \( k \)-free values of polynomials. Suppose \( f(x) \in \mathbb{Z}[x] \) is irreducible and \( \deg f = n \). In what follows, we suppose further that \( f \) has no fixed \( k \)-th power divisors.

**Theorem (Nair):** Let \( k \geq n + 1 \). For \( x \) sufficiently large, there is an integer \( m \) such that \( f(m) \) is \( k \)-free with

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x < m \leq x + cx^{\frac{n}{2k-n+1}}.
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where $r = \sqrt{2n} - \frac{1}{2}$. 
Basic Idea: One works in a number field where $f(x)$ has a linear factor. As in the case $f(x) = x$, one wants to show certain $u$ (in the ring of algebraic integers in the field) are not close by considering

$$(u + a)^k P - u^k Q$$

arising from Padé approximations. One uses that this expression is an integer and, hence, either 0 or $\geq 1$. 
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**Solution:** If it’s small, work with a conjugate instead.
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Theorem: If $f(x, y)$ is an irreducible binary form of degree $n$ and $k \geq (2\sqrt{2} - 1)n/4$, then there are infinitely many integer pairs $(a, b)$ for which $f(a, b)$ is $k$-free.
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The $abc$-Conjecture: For $a$ and $b$ in $\mathbb{Z}^+$, define

$$L_{a,b} = \frac{\log(a + b)}{\log Q(ab(a + b))}$$

and

$$\mathcal{L} = \{ L_{a,b} : a \geq 1, b \geq 1, \gcd(a, b) = 1 \}.$$ 

The set of limit points of $\mathcal{L}$ is the interval $[1/3, 1]$. 
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**Theorem:** The set of limit points of \( \mathcal{L} \) includes the interval \([1/3, 36/37]\).
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(work of Browkin, Greaves, F., Nitaj, Schinzel)
**Approach:** Makes use of a preliminary result about square-free values of binary forms.


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\[
f(x, y) = x y (x + y) (x - y) (x^2 + y^2) (2x^2 + y^2) (x^2 + 2y^2) \\
\times (x^4 - x^2 y^2 + y^4) (3x^4 + 3x^2 y^2 + y^4) (x^4 + 3x^2 y^2 + 3y^4)
\]

the number \( f(x, y)/6 \) takes on the right proportion of squarefree values for

\[
X < x \leq 2X, \quad Y < y \leq 2Y, \quad X = Y^\alpha,
\]

where \( \alpha \in (1, 3) \).
Polynomial Identity:
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\[ P_3(z) - (1 - z)^7 Q_3(z) = z^7 E_3(z) \]

where

\[ P_3(z) = (2z - 1)(3z^2 - 3z + 1), \]
\[ Q_3(z) = -(z + 1)(z^2 + z + 1), \]

and

\[ E_3(z) = -(z - 2)(z^2 - 3z + 3) \]
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\[ z = \frac{x}{x + y} \implies \left\{ \begin{array}{l}
(x + y)^7 (x - y) (x^2 - xy + y^2) \\
+ y^7 (2x + y) (3x^2 + 3xy + y^2) \\
= x^7 (x + 2y) (x^2 + 3xy + 3y^2)
\end{array} \right. \]
\[(x + y)^7(x - y)(x^2 - xy + y^2) + y^7(2x + y)(3x^2 + 3xy + y^2) = x^7(x + 2y)(x^2 + 3xy + 3y^2)\]
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\[f(x, y) = xy(x + y)(x - y)(x^2 + y^2)(2x^2 + y^2)(x^2 + 2y^2) \times (x^4 - x^2y^2 + y^4)(3x^4 + 3x^2y^2 + y^4)(x^4 + 3x^2y^2 + 3y^4)\]
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\[ L_{a,b} = \frac{\log(a + b)}{\log Q(ab(a + b))} \approx \frac{20\alpha \log Y}{\log 20} \]
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\[ L_{a,b} = \frac{\log(a + b)}{\log Q(ab(a + b))} \approx \frac{20\alpha \log Y}{(21\alpha + 1) \log Y} \]
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\[ 1 < \alpha < 3 \implies \frac{10}{11} < L_{a,b} < \frac{15}{16} \]
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**Comment:** This shows \([10/11, 15/16]\) is contained in the set of limit points of \(L_{a,b}\). A similar argument is given for other subintervals of \([1/3, 36/37]\) (not all involving Padé approximations).