Lecture 9: The Polynomial $f(x)x^n + g(x)$

**Theorem (Schinzel; F., Ford, Konyagin):** Let $f(x)$ and $g(x)$ be in $\mathbb{Z}[x]$ with $f(0) \neq 0$, $g(0) \neq 0$, and $\gcd_z(f(x), g(x)) = 1$. Let $r_1$ and $r_2$ denote the number of non-zero terms in $f(x)$ and $g(x)$, respectively. If

$$n \geq 2 \max \left\{ 5^{2N-1}, \max \{ \deg f, \deg g \} \left( 5^{N-1} + \frac{1}{4} \right) \right\} \text{ with } N = 2(\|f\|^2 + \|g\|^2 + r_1 + r_2) - 7,$$

then the non-reciprocal part of $f(x)x^n + g(x)$ is irreducible or identically $\pm 1$ unless one of the following holds:

(i) The polynomial $-f(x)g(x)$ is a $p$th power for some prime $p$ dividing $n$.
(ii) For either $\varepsilon = 1$ or $\varepsilon = -1$, one of $\varepsilon f(x)$ and $\varepsilon g(x)$ is a 4th power, the other is 4 times a 4th power, and $n$ is divisible by 4.

**Notation:** For $n$ a positive integer, $a \mod n$ is the unique integer $b$ in $\{0, 1, \ldots, n-1\}$ such that $a \equiv b \pmod{n}$. Also, $\|x\|$ is the distance from $x$ to its nearest integer.

**Congruence Problem:** Let $a_1, a_2, \ldots, a_r$ denote distinct non-negative integers written in increasing order. Determine an integer $k \geq 2$ such that $a_j \mod k \in [0, k/4) \cup (3k/4, k)$ for each $j \in \{1, 2, \ldots, r\}$.

**Comment:** Clearly, $\exists k \leq 4a_r + 1$. For $\{5, 8\}$ and $\{20, 75, 138\}$, this is the minimum such $k$.

**Actual Problem:** Show that if $a_r$ is large as a function of $r$, then the minimum such $k$ is “small” (smaller than $4a_r + 1$).

**Lemma:** Let $r$ be a positive integer, and let $k_0$ be a real number $\geq 2$. Set

$$A(r) = \max \left\{ 2 \times 5^{2r-1}, k_0 \left( 5^{r-1} + \frac{1}{4} \right) \right\}.$$

Let $a_1, a_2, \ldots, a_r$ be non-negative integers satisfying $a_1 < a_2 < \cdots < a_r$ and $a_r \geq A(r)$. Then there exists an integer $k \in [k_0, 4a_r/3)$ such that $a_j \mod k$ is in $[0, k/4) \cup (3k/4, k)$ for each $j$.

**Main Ideas for Proof of Lemma:**

- Define $x_j = a_j/a_r$ for $j \in \{1, 2, \ldots, r\}$.
- By the Dirichlet box principle, there is an integer $d$ satisfying $1 \leq d \leq 5^{r-1}$ and $\|dx_j\| \leq 1/5$ for $1 \leq j \leq r - 1$. Note that the same inequality holds for $j = r$.
- Since $a_r \geq 2 \times 5^{2r-1}$, we have $d \leq \sqrt{a_r/10}$.
- For $1 \leq j \leq r$, let $c_j$ denote the nearest integer to $dx_j$.
- First, suppose $c_j \neq 0$ (so that $c_j \geq 1$) for each $j \in \{1, 2, \ldots, r\}$. 

Main Ideas for Proof of Theorem:

- For each \( j \in \{1, 2, \ldots, r\} \), \( c_j \leq d \) implies
  \[
  \frac{d + (1/5)}{d + (1/4)} \geq \frac{c_j + (1/5)}{c_j + (1/4)} \geq \frac{dx_j}{c_j} \quad \text{and} \quad \frac{d - (1/5)}{d - (1/4)} \leq \frac{c_j - (1/5)}{c_j - (1/4)} \leq \frac{dx_j}{c_j}.
  \]

- If \( \frac{k}{a_r} \in \left( \frac{d + (1/5)}{d(d + (1/4))}, \frac{d - (1/5)}{d(d - (1/4))} \right) \subseteq \bigcap_{1 \leq j \leq r} \left( \frac{x_j}{c_j + (1/4)}, \frac{x_j}{c_j - (1/4)} \right) \), then \( |a_j - c_jk| < k/4 \) for \( 1 \leq j \leq r \).

- The first interval above has length \( > 1/(10d^2) \geq 1/a_r \), so \( k \) exists. Justify \( k_0 \leq k < 4a_r/3 \).

- If some \( c_j = 0 \), again choose \( k \) in the interval above. Now, \( c_j = 0 \) implies \( |a_j| < k/4 \) since \( 5da_j \leq a_r < kd(d + \frac{1}{3})/(d + \frac{1}{3}) \leq 5dk/4 \).

Main Ideas for Proof of Theorem:

- Assume the non-reciprocal part of \( F(x) = f(x)x^n + g(x) = \sum_{j=0}^r a_jx^{d_j} \) is reducible. Then there are non-reciprocal polynomials \( u(x) \) and \( v(x) \) in \( \mathbb{Z}[x] \) such that \( F(x) = u(x)v(x) \).

- Define \( W(x) = u(x)\tilde{v}(x) = \sum_{j=0}^s b_jx^{e_j} \) and note that \( F(x)\tilde{F}(x) = W(x)\tilde{W}(x) \) and \( \|W\|^2 = \|F\|^2 \) (so \( s \leq \|F\| - 1 \)).

- Define \( T = \{d_1, d_2, \ldots, d_r\} \cup \{d_r - d_1, d_r - d_2, \ldots, d_r - d_r - 1\} \cup \{e_1, e_2, \ldots, e_{s-1}\} \cup \{e_s - 1, e_s - e_2, \ldots, e_s - e_{s-1}\} \). Thus, \( |T| \leq 2 \|F\|^2 + 2r - 5 \).

- Let \( k_0 = 2 \max\{\deg f, \deg g\} \). By the lemma, there is a \( k \in [k_0, 4d_r/3] \) such that \( t \mod k \) is in \( [0, k/4) \cup (3k/4, k) \) for each \( t \in T \).

- Define \( \tilde{d}_j \) and \( \ell_j \) by \( \tilde{d}_j = (d_j + [k/4]) \mod k \) and \( d_j + [k/4] = k\ell_j + \tilde{d}_j \).

- Set \( G_1(x, y) = \sum_{j=0}^r a_jx^{d_j}y^{\ell_j} \) so that \( G_1(x, x^k) = x^{[k/4]}F(x) \). Similarly, define \( G_2(x, y), H_1(x, y), \) and \( H_2(x, y) \) so that \( G_2(x, x^k) = x^{[k/4]}\tilde{F}(x), H_1(x, x^k) = x^{[k/4]}W(x) \), and \( H_2(x, x^k) = x^{[k/4]}\tilde{W}(x) \).

- Writing \( G_1(x, y)G_2(x, y) = \sum_{j=0}^r g_j(x)y^j \), we deduce here that the terms in \( g_j(x) \) correspond precisely to the terms in the expansion of \( x^{2(k/4)}F(x)\tilde{F}(x) \) having degrees in the interval \( [kj, k(j + 1)) \). A similar conclusion holds for the terms in \( H_1(x, y)H_2(x, y) \).

- Deduce \( G_1(x, y)G_2(x, y) = H_1(x, y)H_2(x, y) \) and, consequently, \( G_1(x, y) \) has a non-trivial irreducible factor other than \( x \).

- Define \( \rho \) by \( g(x) = \sum_{j=0}^\rho a_jx^{d_j} \) and \( f(x) = \sum_{j=\rho+1}^r a_jx^{d_j-n} \).

- Observe that \( \ell_0 = \ell_1 = \cdots = \ell_\rho = 0 \) (since \( d_j + [k/4] < k \) for \( j \in \{0, 1, \ldots, \rho\} \) and, for some \( \ell, \ell_{\rho+1} = \ell_{\rho+2} = \cdots = \ell_r = \ell \) (otherwise, \( d_r - d_{\rho+1} \geq \deg f \)).

- Deduce \( G_1(x, y) = f(x)x^{d_y}y^\ell + g(x)x^d \) for some positive integer \( \ell \) and some non-negative integers \( d \) and \( d' \).

- Apply Capelli’s theorem.