Lecture 4: Ljunggren’s Approach to Specific Lacunary Results

Theorem (Ljunggren): Let $n$ and $m$ be integers with $n > m > 0$, and let $\varepsilon_j \in \{1, -1\}$ for $j \in \{1, 2\}$. Then the non-cyclotomic part of $x^n + \varepsilon_1 x^m + \varepsilon_2$ is irreducible or identically 1.

Proof:

- The non-reciprocal part of $f(x) = x^n + \varepsilon_1 x^m + \varepsilon_2$ is the same as the non-cyclotomic part of $f(x)$ (consider $\varepsilon_2 f(x) - \tilde{f}(x)$).
- Suppose $w(x) \in \mathbb{Z}[x]$ with $w(x)w(x) = f(x)\tilde{f}(x)$. The goal is to show $w(x) = \pm f(x)$ or $w(x) = \pm \tilde{f}(x)$. This will imply the non-reciprocal (equivalently, non-cyclotomic) part of $f(x)$ is irreducible or 1.
- We can suppose $w(x)$ has positive leading coefficient and $m \leq n - m$ (the latter by using $\tilde{f}$ instead of $f$ if necessary).
- Observe that $w(0) \neq 0$ and $w(x), \tilde{w}(x), f(x)$, and $\tilde{f}(x)$ have the same degree, namely $n$.
- Since $\|w\|^2 = \|f\|^2 = 3$, each coefficient of $w(x)$ is either 1 or $-1$. Write $w(x) = x^n + \varepsilon_1 x^m + \varepsilon_2$ where $\varepsilon_j \in \{1, -1\}$.
- We can suppose $k \leq n - k$.
- Note that

\[
\begin{align*}
 f(x)\tilde{f}(x) &= (x^n + \varepsilon_1 x^m + \varepsilon_2)(\varepsilon_2 x^n + \varepsilon_1 x^{n-m} + 1) \\
 &= \varepsilon_2 + \varepsilon_1 x^m + \varepsilon_1 \varepsilon_2 x^{n-m} + 3x^n + \varepsilon_1 \varepsilon_2 x^{n+m} + \varepsilon_1 x^{2n-m} + \varepsilon_2 x^{2n}
\end{align*}
\]

and

\[
\begin{align*}
 w(x)\tilde{w}(x) &= (x^n + \varepsilon_1 x^m + \varepsilon_2)(\varepsilon_1' x^n + \varepsilon_2' x^{n-k} + 1) \\
 &= \varepsilon_1' + \varepsilon_1 x^m + \varepsilon_1 \varepsilon_1' x^{n-k} + 3x^n + \varepsilon_1 \varepsilon_1' x^{n+k} + \varepsilon_1' x^{2n-k} + \varepsilon_2' x^{2n}.
\end{align*}
\]

- Comparing the least two exponents above, $\varepsilon_1' = \varepsilon_2, \varepsilon_2' = \varepsilon_1$, and $k = m$. Thus, $w(x) = f(x)$.

Theorem (F. & Solan): Let $f(x) = x^n + x^m + x^p + x^q + 1$ be a polynomial with $n > m > p > q > 0$. Then the non-reciprocal part of $f(x)$ is either irreducible or 1.

Proof:

- Suppose $w(x) \in \mathbb{Z}[x]$ with $w(x)\tilde{w}(x) = f(x)\tilde{f}(x)$. The goal is to show $w(x) = \pm f(x)$ or $w(x) = \pm \tilde{f}(x)$.
- In this case, we may further suppose $w(x)$ is a 0, 1-polynomial (and do so). Write $w(x) = x^n + x^{k_3} + x^{k_2} + x^{k_1} + 1$ with $0 < k_1 < k_2 < k_3 < n$.
- By considering reciprocal polynomials if necessary, we consider $m+q \leq n$ and $k_1 + k_3 \leq n$. 

The condition \( w(x) \tilde{w}(x) = f(x) \tilde{f}(x) \) implies
\[
x^{2n} + x^{2n-q} + x^{2n-p} + x^{2n-m} + x^{n+m} \\
+ x^{n+p} + x^{n+q} + x^{n+m-q} + x^{n+p-q} + 5x^n + \ldots
\]
\[
= x^{2n} + x^{2n-k_1} + x^{2n-k_2} + x^{2n-k_3} + x^{n+k_3} + x^{n+k_2} \\
+ x^{n+k_1} + x^{n+k_3-k_1} + x^{n+k_2-k_1} + x^{n+k_2-k_1} + 5x^n + \ldots.
\]

- Deduce \( 2n - k_1 = 2n - q \) so that \( k_1 = q \).

- By adding exponents, deduce \( 14n + 2k_3 - 2k_1 = 14n + 2m - 2q \) so \( k_3 = m \).

- Substitute and compare exponents to obtain
\[
\{2n - p, n + p, n + m - p, n + p - q\} = \{2n - k_2, n + k_2, n + k_3 - k_2, n + k_2 - k_1\}.
\]

- Comparing largest elements of these sets, deduce one of \( 2n - p \) and \( n + p \) must equal one of \( 2n - k_2 \) and \( n + k_2 \).

- If \( 2n - p = 2n - k_2 \) or \( n + p = n + k_2, k_2 = p \) and \( w(x) = f(x) \).

- If \( 2n - p = n + k_2 \) or \( n + p = 2n - k_2 \), then \( k_2 = n - p \). Substituting and comparing exponents, deduce
\[
\{n + m - p, n + p - q\} = \{n + k_3 - k_2, n + k_2 - k_1\} = \{m + p, 2n - p - q\}.
\]

If \( n + m - p = m + p \), then \( n = 2p \) so that \( k_2 = n - p = p \) and \( w(x) = f(x) \). If \( n + m - p = 2n - p - q \), then \( n = m + q \) so that \( k_3 = m = n - q, k_1 = q = n - m \), and \( w(x) = \tilde{f}(x) \).