ON THE FACTORIZATION OF $n(n + 1)$

by Michael Filaseta

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Joint Work with M. Bennett & O. Trifonov
Part I: On the factorization of $x^2 + x$
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for some integer \( m \), then \( m > 1 \).
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**Effective Approach:** (Linear Forms of Logarithms)

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**Problem:** Can we narrow the gap between these ineffective and effective results?
Don’t Get Me Started:
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Theorem (R. Gow, 1989): If $n > 2$ is even and

$$L_{n}^{(n)}(x) = \sum_{j=0}^{n} \binom{2n}{n-j} \frac{(-x)^j}{j!}$$

is irreducible, then the Galois group of $L_{n}^{(n)}(x)$ is $A_n$. 
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**Theorem (joint work with R. Williams):** For almost all positive integers \( n \) the polynomial \( L_n^{(n)}(x) \) is irreducible (and, hence, has Galois group \( A_n \) for almost all even \( n \)).

**Work in Progress with Trifonov:** We’re attempting to show the irreducibility of \( L_n^{(n)}(x) \) for all \( n > 2 \).
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Theorem: If $n \geq 9$ and
\[ n(n + 1) = 2^k3^\ell m, \]
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Theorem: If $n \geq 9$ and

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$$m \geq n^{1/4}.$$
Part II: On the non-factorization of $x^2 + 7$
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**Classical Ramanujan-Nagell Theorem:** If $x$ and $n$ are integers satisfying

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\[
\left( \frac{x + \sqrt{-7}}{2} \right) \left( \frac{x - \sqrt{-7}}{2} \right) = \left( \frac{1 + \sqrt{-7}}{2} \right)^{n-2} \left( \frac{1 - \sqrt{-7}}{2} \right)^{n-2} m
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Theorem: If $x$, $n$ and $m$ are positive integers satisfying

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then

\[ m \geq ??? \]
Theorem: If $x, n$ and $m$ are positive integers satisfying
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then
\[ m \geq x^{0.4345}. \]
Part III: The Method
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**Main Idea:** Find “small” integers \( P, Q, \) and \( E \) such that

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Obtain an upper bound on $3^k$. Since $3^k m_1 \geq n$, it follows that $m_1$ and, hence, $m = m_1 m_2$ are not small.
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Obtain an upper bound on $3^k$. Since $3^k m_1 \geq n$, it follows that $m_1$ and, hence, $m = m_1 m_2$ are not small.

Use Padé approximations for $(1 - z)^k$ to obtain $P$, $Q$, and $E$. 

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In the case of $x^2 + 7 = 2^n m$, the difference of the primes $(1 + \sqrt{-7})/2$ and $(1 - \sqrt{-7})/2$ each raised to the 13th power has absolute value $\approx 2.65$ and the prime powers themselves have absolute value $\approx 90.51$. 