SQUAREFREE VALUES OF POLYNOMIALS ALL OF WHOSE COEFFICIENTS ARE 0 AND 1

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1. Introduction

Let n be a non-negative integer and consider the set of polynomials

$$S_n = \{f(x) = \sum_{j=0}^{n} \varepsilon_j x^j : \varepsilon_j \in \{0,1\} \text{ for each } j \text{ and } \varepsilon_0 = 1\}.$$

The condition $\varepsilon_0 = 1$ ensures that the elements of S_n are not divisible by x. Let

$$S = \bigcup_{n=0}^{\infty} S_n.$$

There are interesting open problems concerning the polynomials in S. Using the main result in [1] (with base 2) or using the well-known explicit formula for the number of irreducible polynomials of degree $\le n$ modulo 2, one can easily show that there are at least on the order of $2^n/n$ irreducible polynomials in S_n. Odlyzko (private communication) has asked whether almost all polynomials in S are irreducible? In other words, does

$$\lim_{n\to\infty} \frac{|\{f(x) \in S_n : f(x) \text{ is irreducible}\}|}{2^n} = 1?$$

It is not even known how to establish that the limit (or the limit supremum) is positive. Another open problem, posed by Odlyzko and Poonen [2], is to determine whether it is true that if α is a root with multiplicity > 1 of some polynomial $f(x)$ in S, then α is a root of unity.

The purpose of this paper is to establish two results concerning the polynomials in S. First, we shall show

Theorem 1. *Let $b = 3$, 4, or 5. Then there are infinitely many polynomials $f(x) \in S$ for which $f(b)$ is squarefree. Moreover, for such b, the density of polynomials $f(x) \in S$ for which $f(b)$ is squarefree is*

$$\lim_{n\to\infty} \frac{|\{f(x) \in S_n : f(b) \text{ is squarefree}\}|}{2^n} = \frac{6}{\pi^2} \prod_{p|b} \left(1 - \frac{1}{p^2}\right)^{-1}. \tag{1}$$

There are other trivial values of b for which one can obtain similar results (when $|b| \le 2$), but we do not know how to establish the analogous results for $b \ge 6$. As an immediate consequence of Theorem 1, we deduce the

Corollary. *Let $b = 3$, 4, or 5. There are infinitely many squarefree numbers in base b consisting only of the digits 0 and 1.*

The arguments can be modified slightly to allow for the possibility that $\varepsilon_0 = 0$ in the definition of S_n. Thus, for $b = 3$, 4, or 5, we can obtain the density of squarefree numbers in base b among the positive integers consisting only of the digits 0 and 1 in base b. For $b = 4$, the density is $1/2$ times the expression on the right-hand side of (1); for $b = 3$ and 5, the density is $3/4$ times the expression on the right-hand side of (1).

It is of some interest to know a corresponding result for base 10. By applying an argument similar to what we will use for $b = 4$ in Theorem 1, it can be shown that there are infinitely many squarefree numbers which consist only of the digits 0, 1, and 2. In fact, if d_1, d_2, and d_3 are any three distinct digits not equal to 0, 4, and 8 in some order, then there are infinitely many squarefree numbers m in base 10 with each digit of m being either d_1, d_2, or d_3. We will not address this issue further here.

Our second theorem concerns squarefree polynomials in S (polynomials without any roots having multiplicity > 1). We shall see how to obtain the next result as a fairly direct consequence of our approach to establishing Theorem 1.

Theorem 2. *Almost all polynomials in S are squarefree. In other words,*

$$\lim_{n\to\infty} \frac{|\{f(x) \in S_n : f(x) \text{ is squarefree}\}|}{2^n} = 1.$$

In the next section, we give a proof of Theorem 1 for the case that $b = 3$. In the process, we will establish some preliminaries for the cases $b = 4$ and 5. The remainder of the proof of Theorem 1 is given in Section 3. In Section 4, we will establish Theorem 2 using a lemma (Lemma 9) which aided in the proof of Theorem 1.

2. Some Preliminaries and the Case $b = 3$

Let n be a positive integer. For integers b and m with $m \geq 2$, we define $t(n) = t(n, m, b)$ as the number of $f(x) \in S_n$ for which m divides $f(b)$. We begin with an estimate for $t(n)$. Suppose first that m and b are integers which are not relatively prime. Then there is a prime p which divides both m and b. Observe that for every $f(x) \in S_n$, we have $f(b) \equiv 1 \pmod{p}$. Hence, for every $f(x) \in S_n$, m does not divide $f(b)$, and we deduce that $t(n) = 0$. The next lemma deals with the remaining situation where m and b are relatively prime integers.

Lemma 1. *Let m and b be relatively prime integers with $m \geq 2$. Then*

$$t(n) = \frac{2^n}{m}\,(1 + o(1))$$

as n approaches infinity.

Proof. Since

$$\sum_{j=0}^{m-1} e^{2\pi i a j/m} = \begin{cases} m & \text{if } m|a \\ 0 & \text{otherwise,} \end{cases}$$

we obtain

$$t(n) = \frac{1}{m} \sum_{f(x)\in S_n} \sum_{j=0}^{m-1} e^{2\pi i f(b) j/m} = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{f(x)\in S_n} e^{2\pi i f(b) j/m}.$$

On the other hand, from the definition of S_n, we have

$$\sum_{f(x)\in S_n} e^{2\pi i f(b)j/m} = e^{2\pi i j/m} \prod_{k=1}^{n} \left(1 + e^{2\pi i b^k j/m}\right).$$

Observe that when $j = 0$, the right-hand side is 2^n. Hence,

$$t(n) = \frac{2^n}{m} + E,$$

where

$$E = \frac{1}{m} \sum_{j=1}^{m-1} e^{2\pi i j/m} \prod_{k=1}^{n} \left(1 + e^{2\pi i b^k j/m}\right).$$

It remains to show that $E = o(2^n)$.

For each $j \in \{1, 2, \ldots, m-1\}$, we rewrite the absolute value of the product above as

$$\begin{aligned} \left|\prod_{k=1}^{n} \left(1 + e^{2\pi i b^k j/m}\right)\right| &= \left|\prod_{k=1}^{n} e^{\pi i b^k j/m}\right| \left|\prod_{k=1}^{n} \left(e^{\pi i b^k j/m} + e^{-\pi i b^k j/m}\right)\right| \\ &= 2^n \prod_{k=1}^{n} \left|\cos(\pi b^k j/m)\right|. \end{aligned}$$

Since m and b are relatively prime and $1 \le j \le m-1$, the expression $b^k j/m$ is a rational number which differs from an integer by at least $1/m$. Therefore,

$$\left|\cos(\pi b^k j/m)\right| \le \left|\cos(\pi/m)\right|.$$

Since $m \ge 2$, this last expression is < 1. We obtain

$$\begin{aligned} |E| &\le \frac{1}{m} \sum_{j=1}^{m-1} \left|\prod_{k=1}^{n} \left(1 + e^{2\pi i b^k j/m}\right)\right| \\ &= \frac{2^n}{m} \sum_{j=1}^{m-1} \prod_{k=1}^{n} \left|\cos(\pi b^k j/m)\right| \le 2^n \left|\cos(\pi/m)\right|^n, \end{aligned}$$

and the lemma easily follows. ■

Lemma 2. *Let b be a positive integer, and let B be a real number > 0. Denote by $S(B,n)$ the number of $f(x) \in S_n$ such that $f(b)$ is not divisible by p^2 for every prime $p \le B$. Then*

$$S(B,n) = 2^n \prod_{p \le B,\ p \nmid b} \left(1 - \frac{1}{p^2}\right) + o\left(2^n\right).$$

Lemma 2 follows from Lemma 1 by an easy sieve argument and we omit the details. Observe that

$$\prod_{p \le B,\ p \nmid b} \left(1 - \frac{1}{p^2}\right) = \prod_{p \nmid b} \left(1 - \frac{1}{p^2}\right)(1 + O(1/B)) = \frac{6}{\pi^2} \prod_{p \mid b} \left(1 - \frac{1}{p^2}\right)^{-1} (1 + O(1/B)).$$

Fix $\varepsilon > 0$. By choosing B sufficiently large and then choosing n sufficiently large, we deduce from Lemma 2 that $S(B,n)$ differs from

$$\frac{6 \times 2^n}{\pi^2} \prod_{p \mid b} \left(1 - \frac{1}{p^2}\right)^{-1}$$

by $\le \varepsilon 2^n$. Thus, to prove Theorem 1, it suffices to show that the number of $f(x) \in S_n$ such that $f(b)$ is divisible by p^2 for some prime $p > B$ is $\le \varepsilon 2^n$. For such an estimate we may suppose that B is arbitrarily large; more specifically, we can take $B \ge B_0$ where B_0 is an arbitrary constant depending only on ε. The proof of Theorem 1 for the case $b = 3$ therefore follows from the following lemma.

Lemma 3. *Let $\varepsilon > 0$, and let B be sufficiently large. Then there are $\le \varepsilon 2^n$ polynomials $f(x) \in S_n$ for which there exists an integer $d > B$ such that $d^2 \mid f(3)$.*

Proof. Let d be an integer $> B$. Let r be the positive integer satisfying

$$3^{r/2} < d \le 3^{(r+1)/2}.$$

We fix $\varepsilon_r, \varepsilon_{r+1}, \ldots, \varepsilon_n \in \{0,1\}$ arbitrarily and consider $f(x) = \sum_{j=0}^{n} \varepsilon_j x^j \in S_n$. Observe that for any choice of $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-1} \in \{0,1\}$, we have

$$0 \le \sum_{j=0}^{r-1} \varepsilon_j 3^j < d^2.$$

Also, for distinct choices of the $r-$tuple $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-1})$ with each $\varepsilon_j \in \{0,1\}$, the numbers $\sum_{j=0}^{r-1} \varepsilon_j 3^j$ are distinct; hence, they are distinct modulo d^2. We deduce that with $\varepsilon_r, \varepsilon_{r+1}, \ldots, \varepsilon_n \in \{0,1\}$ fixed, there is at most one choice of $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-1})$ such that $f(3)$ is divisible by d^2. It follows that there are at most 2^{n-r+1} choices for $f(x) \in S_n$ such that $f(3)$ is divisible by d^2. The inequality $3^{(r+1)/2} \geq d > B$ implies that r is large. Hence,

$$2^{n-r+1} = 2^{n+1}2^{-r} = 2^{n+1}(3^{r/2})^{-2\log 2/\log 3} \leq 2^{n+1}(3^{(r+1)/2})^{-5/4} \leq 2^{n+1}d^{-5/4}.$$

We deduce that the number of $f(x) \in S_n$ such that $f(3)$ is divisible by d^2 for some integer $d > B$ is

$$\leq 2^{n+1} \sum_{d>B} d^{-5/4}.$$

Since B is sufficiently large and $\sum_{d=1}^{\infty} d^{-5/4}$ converges, we deduce that this last expression is $\leq \varepsilon 2^n$, completing the proof of the lemma. ■

3. The Cases $b = 4$ and $b = 5$

In this section, we complete the proof of Theorem 1. We will improve on the argument given for Lemma 3 to obtain the desired result. We note that the work in this section allows us also to handle the case $b = 3$ here, but we have chosen to indicate the proof of the case $b = 3$ separately in the previous section partially because of its simplicity and partially because the case $b = 3$ of Theorem 1 by itself can be used to obtain Theorem 2 (see Section 4).

As in the previous section, we fix $\varepsilon > 0$ and consider B to be sufficiently large. Analogous to Lemma 3, we want to show for $b = 4$ and $b = 5$ that the number of $f(x) \in S_n$ such that $f(b)$ is divisible by d^2 for some $d > B$ is $\leq \varepsilon 2^n$.

For $b \geq 3$, we define

$$S(b) = \left\{ \sum_{j=0}^{\infty} \varepsilon_j b^j : \varepsilon_j \in \{0,1\},\ \text{all but finitely many } \varepsilon_j \text{ are } 0 \right\}$$

and

$$\begin{aligned} S' = S'(b) &= \{m_1 - m_2 : m_1, m_2 \in S(b), m_1 > m_2\} \\ &= \left\{ \sum_{j=0}^{\infty} \varepsilon_j b^j \in \mathbb{Z}^+ : \varepsilon_j \in \{-1,0,1\},\ \text{all but finitely many } \varepsilon_j \text{ are } 0 \right\}. \end{aligned}$$

For r and t positive integers, we consider the set

$$X(r,t) = X(r,t;b) = \{u \in \mathbb{Z} \cap [b^{r-1}, b^r) : \gcd(b,u) = 1 \text{ and } tu^2 \in S'\}.$$

The next several lemmas serve to estimate the size of $X(r,t)$. In the end, we will need a more intricate estimate for the case $b = 5$ than for the case $b = 4$; in particular, for the case $b = 5$, we will need to strengthen our next lemma which is a preliminary bound on $|X(r,t)|$.

Lemma 4. *Let $b \geq 3$, $r \geq 2$, and $t \geq 1$ be integers. Then*

$$|X(r,t)| \leq 3^{r+1} b^2.$$

Proof. For any positive integers m and s, m is in S' if and only if $b^s m$ is in S'. Thus, we may suppose that $b \nmid t$, and we do so. We may also suppose that $|X(r,t)| \neq 0$. Let u be in $X(r,t)$. Then tu^2 is in S'. By the definition of S', an element of S' is either relatively prime to b or it is divisible by b. Thus, the conditions $\gcd(b,u) = 1$ and $tu^2 \in S'$ imply $\gcd(b,t) = 1$.

We write

$$tu^2 = \sum_{k=0}^{\infty} \alpha_k b^k,$$

where each $\alpha_k = \alpha_k(u)$ is in $\{-1, 0, 1\}$. There are $3^{r+1}b^2$ different values for the $(r+2)-$tuple $(u', \alpha'_0, \alpha'_1, \ldots, \alpha'_r)$ where u' is a non-negative integer $< b^2$ and $\alpha'_k \in \{-1, 0, 1\}$ for $k \in \{0, 1, \ldots, r\}$. Consider a fixed such $(r+2)-$tuple. The lemma will follow if we can show that there is at most one $u \in X(r,t)$ for which $u \equiv u' \pmod{b^2}$ and $\alpha_k(u) = \alpha'_k$ for every $k \in \{0, 1, \ldots, r\}$.

Let u and v be in $X(r,t)$ with $u \equiv v \pmod{b^2}$ and $\alpha_k(u) = \alpha_k(v)$ for every $k \in \{0, 1, \ldots, r\}$. We want to show that $u = v$. Let p be a prime divisor of b. Then $\gcd(b, u) = 1$ implies $p \nmid u$. Since $\gcd(u - v, u + v) = \gcd(u - v, 2u)$, we deduce that if p divides both $u - v$ and $u + v$, then $p = 2$. Also, $u \equiv v \pmod{b^2}$ implies $p^2 | (u - v)$ so that in the case $p = 2$, we have $4 \nmid (u + v)$. Since $\gcd(b, t) = 1$, it follows that $\gcd(b^{r+1}, t(u + v))$ is either 1 or 2 and, hence, divides b. The condition $\alpha_k(u) = \alpha_k(v)$ for every $k \in \{0, 1, \ldots, r\}$ implies $b^{r+1} | \left(tu^2 - tv^2\right)$. We deduce $b^r | (u - v)$. The conclusion $u = v$ now follows since u and v are positive integers $< b^r$. ■

Lemma 5. *Let j and s be positive integers. Let K be a set of $s-$tuples $(\kappa_1, \ldots, \kappa_s)$ satisfying the two conditions:*

(i) For each $i \in \{1, 2, \ldots, s\}$, $\kappa_i \in \{1, 2, 3\}$.

(ii) For each $i \in \{j+1, j+2, \ldots, s\}$, if $\kappa_{i-j} \in \{2, 3\}$, then $\kappa_i \in \{1, 2\}$.

Then

$$|K| \leq \left(\frac{3}{1 + \sqrt{2}}\right)^j \left(1 + \sqrt{2}\right)^s.$$

Proof. For each $t \in \{1, 2, \ldots, j\}$, consider the elements $(\kappa_1, \ldots, \kappa_s)$ of K and define K_t as the set of $[(s - t + j)/j]-$tuples $(\kappa_t, \kappa_{j+t}, \ldots, \kappa_{[(s-t)/j]j+t})$. Thus, $|K| \leq \prod_{t=1}^{j} |K_t|$. Also, observe that the number of components in each element of K is the sum over t of the number of components in each element of K_t. In other words,

(2) $$s = \sum_{t=1}^{j} \left[\frac{s - t + j}{j}\right].$$

Fixing $t \in \{1, 2, \ldots, j\}$, we consider the elements $(\psi_1, \psi_2, \ldots, \psi_{[(s-t+j)/j]})$ of K_t. For each $i \in \{1, 2, \ldots, [(s-t+j)/j]\}$, we define N_i as the number of different choices for $\psi_1, \psi_2, \ldots, \psi_i$ which arise. In other words, N_i is the number of $i-$tuples $(\psi_1, \psi_2, \ldots, \psi_i)$ obtained from the first i components of the elements of K_t. Thus, $|K_t| = N_{[(s-t+j)/j]}$. By condition (i), $N_1 \le 3$. By conditions (i) and (ii), $N_2 \le 7$ (there are ≤ 3 choices for (ψ_1, ψ_2) with $\psi_1 = 1$ and ≤ 4 choices for (ψ_1, ψ_2) with $\psi_1 \in \{2, 3\}$). Fix $i \in \{3, 4, \ldots, [(s-t+j)/j]\}$. Let M be the number of $(i-1)-$tuples $(\psi_1, \psi_2, \ldots, \psi_{i-1})$ with $\psi_{i-1} = 1$. Observe that $M \le N_{i-2}$. By condition (i), there are $\le 3M$ possible $i-$tuples $(\psi_1, \psi_2, \ldots, \psi_i)$ with $\psi_{i-1} = 1$. On the other hand, by condition (ii), there are $\le 2(N_{i-1} - M)$ possible $i-$tuples $(\psi_1, \psi_2, \ldots, \psi_i)$ with $\psi_{i-1} \in \{2, 3\}$. Therefore,

$$N_i \le 3M + 2(N_{i-1} - M) = 2N_{i-1} + M \le 2N_{i-1} + N_{i-2}.$$

Recall that $N_1 \le 3$ and $N_2 \le 7$. An easy induction argument now gives $N_i \le 3(1+\sqrt{2})^{i-1}$. Thus,

$$|K_t| = N_{[(s-t+j)/j]} \le 3(1+\sqrt{2})^{[(s-t)/j]} = \left(\frac{3}{1+\sqrt{2}}\right)\left(1+\sqrt{2}\right)^{[(s-t+j)/j]}.$$

The lemma now follows from $|K| \le \prod_{t=1}^{j} |K_t|$ and (2). ■

Lemma 6. *Let b be an odd integer ≥ 5, and let r and j be positive integers with $j \le r$. Let a and t be positive integers and suppose that $b^j || a$. Then the number of positive integers $u < b^r$ with $\gcd(b, u) = 1$ and such that both tu^2 and $t(u+a)^2$ are in S' is $\le (b-1)3^j(1+\sqrt{2})^{r-j}$.*

Proof. As in the proof of Lemma 4, we may suppose that $\gcd(b, t) = 1$ and do so. Let u be as in the statement of the lemma. Let

$$D(u) = t(u+a)^2 - tu^2 = ta(2u+a).$$

Since tu^2 and $t(u+a)^2$ are in S', we have

$$(3) \qquad tu^2 = \sum_{k=0}^{\infty} \alpha_k b^k \quad \text{and} \quad t(u+a)^2 = \sum_{k=0}^{\infty} \beta_k b^k$$

for some integers α_k and β_k in $\{-1,0,1\}$. We write

$$(4) \qquad u = \sum_{k=0}^{r-1} u_k b^k \quad \text{and} \quad D(u) = \sum_{k=0}^{\infty} d_k b^k$$

where, for each non-negative integer k, $u_k \in [0, b-1]$ and

$$(5) \qquad d_k = \beta_k - \alpha_k \in [-2, 2].$$

Note that since $b \geq 5$, $D(u)$ has a unique representation as in (4) with $d_k \in [-2,2]$. Suppose now that v is a positive integer $< b^r$ with $v \neq u$ and $\gcd(b,v)=1$ and such that both tv^2 and $t(v+a)^2$ are in S'. Let ℓ be the non-negative integer satisfying $b^\ell || (v-u)$. Then $D(v) - D(u) = 2ta(v-u)$ so that

$$b^{\ell+j} || \left(D(v) - D(u)\right).$$

Viewing the numbers $u_0, u_1, \ldots, u_{\ell-1}$ in (4) as fixed, we deduce that the numbers $d_0, d_1, \ldots, d_{\ell+j-1}$ are uniquely determined. Furthermore, the number u_ℓ uniquely determines the value of $d_{\ell+j}$ and different values of u_ℓ lead to different values of $d_{\ell+j}$. In particular, there is at most one choice of u_ℓ which leads to $d_{\ell+j} = 0$. We refer to such a choice of u_ℓ as "nice."

We keep the notation above and still view $u_0, u_1, \ldots, u_{\ell-1}$ as fixed. Suppose that $\ell \geq 1$. Since b is an odd integer relatively prime to tu, we obtain that $\gcd(b, t(u+v)) = 1$ so that $b^\ell || \left(tv^2 - tu^2\right)$. Hence, the numbers $\alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}$ in (3) are uniquely determined. Different values of u_ℓ lead to different values of α_ℓ. We are interested only in u for which $tu^2 \in S'$ so that $\alpha_\ell \in \{-1,0,1\}$. Therefore, there are at most 3 different values of u_ℓ such that $tu^2 \in S'$.

Since α_ℓ and β_ℓ are in $\{-1,0,1\}$, for each $d_\ell \in \{-2,-1,0,1,2\}$, there are at most $3-|d_\ell|$ values of α_ℓ such that (5) holds. In particular, we deduce that if $\ell \geq j$ and $u_{\ell-j}$ is not nice (so that $d_\ell \neq 0$), then there are at most two values of α_ℓ, and hence at most two values of u_ℓ, for which tu^2 and $t(u+a)^2$ are both in S'.

Since $b \nmid u$, there are at most $b-1$ choices for u_0 in (4). Fix u_0 and consider the choices for $u_1, \ldots, u_{r-1}$ as in (4) with u as in the lemma. For $\ell \in \{1,2,\ldots,r-1\}$ and for any given $u_1,\ldots,u_{\ell-1}$, there are at most 3 different values of u_ℓ, say $\gamma_i = \gamma_i(u_0,u_1,\ldots,u_{\ell-1})$ where i is a positive integer ≤ 3. At most one such u_ℓ is nice, and if such a choice of u_ℓ exists we can suppose that it is γ_1 and do so. We define $\phi_\ell(u_\ell) = i$ where $i \in \{1,2,3\}$ with $u_\ell = \gamma_i$. Observe that u in (4) is uniquely determined by the value of $(\phi_1(u_1), \phi_2(u_2), \ldots, \phi_{r-1}(u_{r-1}))$ (where we are still viewing u_0 as fixed). Also, if $\ell \in \{j+1, j+2, \ldots, r-1\}$ and $\phi_{\ell-j}(u_{\ell-j}) \in \{2,3\}$ (so that $u_{\ell-j}$ is not nice), then $\phi_\ell(u_\ell) \leq 2$. Thus, the set of $(r-1)$–tuples $(\phi_1(u_1), \ldots, \phi_{r-1}(u_{r-1}))$ satisfies the conditions of the set K in Lemma 5 with $s = r-1$. Recalling that there are $\leq b-1$ choices for the value of u_0, we deduce that the number of $u < b^r$ with $\gcd(b,u)=1$ and such that both tu^2 and $t(u+a)^2$ are in S' is

$$\leq (b-1)\left(\frac{3}{1+\sqrt{2}}\right)^j \left(1+\sqrt{2}\right)^{r-1} < (b-1)3^j(1+\sqrt{2})^{r-j},$$

establishing the lemma. ■

Lemma 7. *Let b be a positive integer ≥ 3. Let r and ℓ be positive integers with $1 \leq \ell \leq r$. Let t be a positive integer. Then there exist $3^{r-\ell+2}$ intervals each of length $< 2b^\ell$ with the union of these intervals containing all numbers u for which $b^{r-1} \leq u < b^r$ and $tu^2 \in S'$.*

Proof. Let s be the positive integer satisfying

$$\frac{b^{s-1}}{b-1} < t \leq \frac{b^s}{b-1}.$$

For $u < b^r$ and $tu^2 \in S'$, we obtain

$$tu^2 = \sum_{k=0}^{2r+s-1} \alpha_k b^k \quad \text{for some } \alpha_k \in \{-1, 0, 1\}.$$

Fix α_k for $r + s + \ell - 2 \le k \le 2r + s - 1$. Let

$$\alpha = \sum_{k=r+s+\ell-2}^{2r+s-1} \alpha_k b^k - \sum_{k=0}^{r+s+\ell-3} b^k \quad \text{and} \quad \beta = \sum_{k=r+s+\ell-2}^{2r+s-1} \alpha_k b^k + \sum_{k=0}^{r+s+\ell-3} b^k.$$

For $b^{r-1} \le u < b^r$ and $tu^2 \in S'$, we deduce that tu^2 is in some such $[\alpha, \beta]$ so that $u \in [\gamma, \delta]$ where

$$[\gamma, \delta] = \left[\sqrt{\alpha/t}, \sqrt{\beta/t}\right] \cap \left[b^{r-1}, b^r\right].$$

Observe that

$$\beta - \alpha = 2 \sum_{k=0}^{r+s+\ell-3} b^k < \frac{2b^{r+s+\ell-2}}{b-1}.$$

Therefore,

$$\begin{aligned} \delta - \gamma \le \sqrt{\beta/t} - \sqrt{\alpha/t} \ &= \frac{\beta - \alpha}{t(\sqrt{\beta/t} + \sqrt{\alpha/t})} \\ < \frac{\beta - \alpha}{t\gamma} \le \frac{\beta - \alpha}{tb^{r-1}} &< \frac{2b^{r+s+\ell-2}/(b-1)}{b^{r+s-2}/(b-1)} = 2b^\ell. \end{aligned}$$

Hence, the $3^{r-\ell+2}$ choices for $\alpha_{r+s+\ell-2}, \ldots, \alpha_{2r+s-1}$, each in $\{-1, 0, 1\}$, lead to $3^{r-\ell+2}$ intervals $[\gamma, \delta]$ of length $< 2b^\ell$ satisfying the conditions of the lemma. ■

Since $b \ge 3$, it is not difficult to check that the intervals in the proof of Lemma 7 above are disjoint. On the other hand, it is already clear in the statement of Lemma 7 that we may consider these intervals to be disjoint.

Lemma 8. *Let b be an odd integer ≥ 5. Let r and t be positive integers. Then*

$$|X(r,t)| \ll \exp\left(\frac{\log 3(\log b + \log(1+\sqrt{2}))\, r}{\log(3b)}\right),$$

where the implied constant depends on b but not on r or t.

Proof. Consider an arbitrary positive integer $\ell \le r$. By Lemma 7, $X(r,t)$ is contained in the union of $3^{r-\ell+2}$ disjoint intervals $[\gamma_i, \delta_i]$, with $1 \le i \le 3^{r-\ell+2}$, where each interval is of length $< 2b^\ell$. For each $i \in \{1, 2, \ldots, 3^{r-\ell+2}\}$ and $k \in \{1, 2, \ldots, b-1\}$, we set

$$X_{i,k}(r,t) = \{u \in X(r,t) : u \in [\gamma_i, \delta_i] \text{ and } u \equiv k \pmod{b}\}.$$

Let $n_{i,k} = |X_{i,k}(r,t)|$. Then

$$\begin{aligned}
\sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} \frac{n_{i,k}(n_{i,k}-1)}{2} &= \sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} |\{(u,v) : u \in X_{i,k}(r,t), v \in X_{i,k}(r,t), \text{ and } u < v\}| \\
&= \sum_{\substack{1 \le a < 2b^\ell \\ b|a}} \sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} |\{(u,v) : u \in X_{i,k}(r,t), v \in X_{i,k}(r,t), \text{ and } v-u = a\}| \\
&\le \sum_{\substack{1 \le a < 2b^\ell \\ b|a}} |\{(u,v) : u \in X(r,t), v \in X(r,t), \text{ and } v-u = a\}|.
\end{aligned}$$

From Lemma 6, we now deduce that

$$\begin{aligned}
\sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} \frac{n_{i,k}(n_{i,k}-1)}{2} &\le \sum_{j=1}^{\ell} \sum_{\substack{1 \le a < 2b^\ell \\ b^j \,||\, a}} (b-1)3^j(1+\sqrt{2})^{r-j} \\
&\le \sum_{j=1}^{\ell} 2b^{\ell-j}(b-1)3^j(1+\sqrt{2})^{r-j} \ll b^\ell(1+\sqrt{2})^r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|X(r,t)| = \sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} n_{i,k} &\le \sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} \left(1 + \frac{n_{i,k}(n_{i,k}-1)}{2}\right) \\
&= \sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} 1 \; + \; \sum_{i=1}^{3^{r-\ell+2}} \sum_{k=1}^{b-1} \frac{n_{i,k}(n_{i,k}-1)}{2} \ll 3^{r-\ell} + b^\ell(1+\sqrt{2})^r.
\end{aligned}$$

We choose

$$\ell = \left[\frac{(\log 3 - \log(1+\sqrt{2}))\, r}{\log(3b)} \right] + 1$$

to obtain the lemma. ■

Lemma 9. *Let $b = 4$ or 5. Let $\varepsilon > 0$, and let $B = B(\varepsilon)$ be sufficiently large. Then the number of $f(x) \in S_n$ such that $f(b)$ is divisible by d^2 for some integer $d > B$ is $\leq \varepsilon 2^n$.*

Proof. Since B is sufficiently large, the number of $f(x) \in S_n$ as in the lemma is 0 unless n is also large. We therefore consider n large. Let r be a positive integer for which $b^r > B$. We consider the integers d such that $b^{r-1} \leq d < b^r$. For $f(x) \in S_n$, we have $0 < f(b) \leq b^{n+1}$ so that if $f(b)$ is divisible by d^2 (which is $\geq b^{2r-2}$), then $r \leq (n+3)/2$. We therefore suppose, as we may, that $r \leq (n+3)/2$.

Recall that each $f(x) \in S_n$ has constant term 1 so that if $f(b)$ is divisible by d^2, then $\gcd(b,d) = 1$. If $f(b) = td^2$, then we also have that $1 \leq t = f(b)/d^2 \leq b^{n-2r+3}$ so that $d \in X(r,t)$ for some positive integer $t \leq b^{n-2r+3}$. We use Lemmas 4 and 8 to obtain that the number of $f(x) \in S_n$ for which there exists a $d \in [b^{r-1}, b^r)$ such that $d^2 | f(b)$ is

$$\leq \sum_{t=1}^{b^{n-2r+3}} |X(r,t)| \ll \begin{cases} 4^{n-2r}3^r & \text{for } b = 4 \\ 5^{n-2r} \exp\left(\dfrac{\log 3(\log 5 + \log(1+\sqrt{2}))\, r}{\log 15} \right) & \text{for } b = 5. \end{cases}$$

In either case, if $r > n/(2.4)$, the above expression on the right is easily $\ll 2^n/(nB)$. We restrict our attention now to $r \leq n/(2.4)$. We note that our method for obtaining this bound on r is not the best possible, and it would be easy to replace 2.4 with a larger number; however, 2.4 will be sufficient for what follows.

Let s denote a positive integer $\leq n - 2r$. We consider $f(x) = \sum_{j=0}^{n} \varepsilon_j x^j \in S_n$ with $\varepsilon_{2r+s-2}, \varepsilon_{2r+s-1}, \dots, \varepsilon_n$ fixed elements from $\{0,1\}$. Thus, we obtain 2^{2r+s-3} different values of $f(b)$. Let $N(d)$ denote the number of different $(2r+s-3)$-tuples $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2r+s-3})$, with each $\varepsilon_j \in \{0,1\}$, such that $d^2 | f(b)$. Suppose $N(d) \geq 1$. Consider the $f(x)$ counted by $N(d)$, and let $f_1(x)$ denote the $f(x)$ which minimizes the value of $f(b)$. Then there are $N(d) - 1$ other $f(x)$ counted by $N(d)$ each having the property that $d^2 | f(b)$. For each of these $N(d) - 1$ different $f(x)$, we obtain

$$0 < f(b) - f_1(b) \leq b^{2r+s-2} \leq d^2 b^s.$$

Thus, there are at least $N(d)-1$ different $f(x) \in S_n$ (with $\varepsilon_{2r+s-2}, \varepsilon_{2r+s-1}, \ldots, \varepsilon_n$ fixed) such that $f(b) - f_1(b) = td^2$ for some positive integer $t \le b^s$. Different choices for $f(x)$ give different values for t. We deduce that there are at least $N(d)-1$ different $t \le b^s$ for which $d \in X(r,t)$.

With $\varepsilon_{2r+s-2}, \varepsilon_{2r+s-1}, \ldots, \varepsilon_n$ still fixed, we bound the number of $f(x) \in S_n$ such that there is a $d \in [b^{r-1}, b^r)$ for which $d^2 | f(b)$. This number is

$$\le \sum_{b^{r-1} \le d < b^r} N(d) = \sum_{\substack{b^{r-1} \le d < b^r \\ N(d) \ge 1}} (N(d)-1) + \sum_{\substack{b^{r-1} \le d < b^r \\ N(d) \ge 1}} 1.$$

From our comments above and from Lemmas 4 and 8, we deduce that

$$\sum_{\substack{b^{r-1} \le d < b^r \\ N(d) \ge 1}} (N(d)-1) \le \sum_{b^{r-1} \le d < b^r} \sum_{\substack{1 \le t \le b^s \\ d \in X(r,t)}} 1 = \sum_{1 \le t \le b^s} \sum_{\substack{b^{r-1} \le d < b^r \\ d \in X(r,t)}} 1$$

$$= \sum_{1 \le t \le b^s} |X(r,t)| \ll \begin{cases} 4^s 3^r & \text{for } b = 4 \\ 5^s \exp\left(\dfrac{\log 3 (\log 5 + \log(1+\sqrt{2}))\, r}{\log 15} \right) & \text{for } b = 5. \end{cases}$$

Also,

$$\sum_{\substack{b^{r-1} \le d < b^r \\ N(d) \ge 1}} 1 \le b^r.$$

Letting $\varepsilon_{2r+s-2}, \varepsilon_{2r+s-1}, \ldots, \varepsilon_n$ now vary, we deduce that the number of $f(x) \in S_n$ such that there exists a $d \in [b^{r-1}, b^r)$ for which $d^2 | f(b)$ is

$$\ll 2^{n-2r-s} 4^s 3^r + 2^{n-2r-s} 4^r \qquad \text{for b} = 4$$

and

$$\ll 2^{n-2r-s} 5^s \exp\left(\frac{\log 3 (\log 5 + \log(1+\sqrt{2}))\, r}{\log 15} \right) + 2^{n-2r-s} 5^r \qquad \text{for b} = 5.$$

In the case $b = 4$, we choose

$$s = \left[\frac{r \log(4/3)}{\log 4} \right] + 1;$$

and in the case $b = 5$, we choose

$$s = \left[\frac{r}{\log 5} \left(\log 5 - \frac{\log 5 + \log(1+\sqrt{2})}{\log 15} (\log 3) \right) \right] + 1.$$

It is easily checked that since $1 \le r \le n/(2.4)$, in either case the choice of s is a positive integer $\le n - 2r$. We obtain that the number of $f(x) \in S_n$ such that $f(b)$ is divisible by some d^2 with $b^{r-1} \le d < b^r$ is

$$\ll 2^{n-2r-s} b^r \ll \begin{cases} 2^n \exp(-0.14r) & \text{for } b = 4 \\ 2^n \exp(-0.034r) & \text{for } b = 5. \end{cases}$$

In either case, $b = 4$ or $b = 5$, since $e^2 > b$, the above bound is $\ll 2^n e^{-2r/100} \ll 2^n b^{-r/100}$.

Letting r vary over the positive integers for which $b^r > B$, we easily obtain now that the number of $f(x) \in S_n$ such that $f(b)$ is divisible by d^2 for some $d > B$ is $\ll 2^n B^{-1/100}$. Since B is sufficiently large, the proof of the lemma is complete. ■

4. The Proof of Theorem 2

Let R be a fixed real number ≥ 1. We begin by estimating the number of $f(x) \in S_n$ divisible by the square of a non-constant polynomial in $\mathbb{Z}[x]$ of degree $\le R$. We will show that there are $o(2^n)$ such $f(x)$.

Odlyzko and Poonen [2] have obtained extensive results about the roots of polynomials in S_n. For our purposes, it suffices to know that these roots are bounded in absolute value by 2 which is easily established as follows. Let $f(x) \in S_n$, and write $f(x) = \sum_{j=0}^{m} \varepsilon_j x^j$ where $m \le n$, $\varepsilon_j \in \{0, 1\}$ for each j, and $\varepsilon_0 = \varepsilon_m = 1$. If $\alpha \in \mathbb{C}$ and $|\alpha| \ge 2$, then

$$|f(\alpha)| \ge \left| \sum_{j=0}^{m} \varepsilon_j \alpha^j \right| \ge |\alpha|^m - \sum_{j=0}^{m-1} |\alpha|^j = |\alpha|^m - \frac{|\alpha|^m - 1}{|\alpha| - 1}$$

$$= \frac{|\alpha|^{m+1} - 2|\alpha|^m + 1}{|\alpha| - 1} = \frac{(|\alpha| - 2)|\alpha|^m + 1}{|\alpha| - 1} > 0.$$

Thus, $f(\alpha) \neq 0$, and we deduce that all roots of the polynomials in S_n necessarily have absolute value < 2.

Let $g(x) \in \mathbb{Z}[x]$ of degree $r \in [1, R]$, and suppose that $g(x)$ is a factor of some polynomial in S_n. It follows that the roots of $g(x)$ are < 2. Also, since polynomials in S_n are monic, the leading coefficient of $g(x)$ must be ± 1. Since the degree of $g(x)$ is $\leq R$, it follows that each coefficient of $g(x)$ has absolute value less than or equal to the product of 2^R (an upper bound on the absolute value of the product of the roots of $g(x)$) and 2^R (an upper bound on the number of combinations of $r \leq R$ roots taken k at a time where $k \in \{0, 1, \ldots, r\}$). Since the absolute value of the coefficients of $g(x)$ are bounded by 4^R and since $g(x)$ has degree $\leq R$, there are

$$\leq \left(2 \times 4^R + 1\right)^{R+1}$$

different possible values of $g(x)$. To establish what we first set out to show, it suffices then to obtain that for each such $g(x)$, there are $o(2^n)$ different possible $f(x) \in S_n$ divisible by $g(x)^2$.

Fix $g(x)$ as above. Suppose that $f(x) = \sum_{j=0}^{n} \varepsilon_j x^j \in S_n$ is divisible by $g(x)^2$. We consider the set $T_n(f(x))$ consisting of the polynomials $w(x) = \sum_{j=0}^{n} \varepsilon'_j x^j \in S_n$ where there is exactly one $k \in \{1, 2, \ldots, n\}$ for which $\varepsilon'_k \neq \varepsilon_k$. In other words, $w(x) = \sum_{j=0}^{n} \varepsilon'_j x^j \in T_n(f(x))$ if and only if there is a $k \in \{1, 2, \ldots, n\}$ such that $\varepsilon'_\ell = \varepsilon_\ell$ for every $\ell \in \{0, 1, \ldots, n\}$ with $\ell \neq k$ and $\varepsilon'_k = 1 - \varepsilon_k$. Thus, $|T_n(f(x))| = n$. Since $f(x)$ is divisible by $g(x)^2$ and $f(x)$ has constant term 1, it must be the case that $g(x)$ is not divisible by x. If $w(x) = \sum_{j=0}^{n} \varepsilon'_j x^j \in T_n(f(x))$ and $k \in \{1, 2, \ldots, n\}$ with $\varepsilon'_k \neq \varepsilon_k$, then $f(x) - w(x) = \pm x^k$ is not divisible by $g(x)^2$. We deduce that the elements of $T_n(f(x))$ are not divisible by $g(x)^2$.

Now, suppose that $f_1(x)$ and $f_2(x)$ are distinct polynomials in S_n with each divisible by $g(x)^2$. We show that $T_n(f_1(x))$ and $T_n(f_2(x))$ are disjoint. If the sets were not disjoint, then there would be some $w(x)$ which differs from each of $f_1(x)$ and $f_2(x)$ by a power of x. By considering $f_1(x) - f_2(x)$, it follows that for some k and ℓ in $\{1, 2, \dots, n\}$ with $k > \ell$, $x^k \pm x^\ell = x^\ell(x^{k-\ell} \pm 1)$ is divisible by $g(x)^2$. Since the roots of $x^{k-\ell} \pm 1$ are distinct and since $g(x)$ is not divisible by x, we deduce that $g(x)^2$ cannot divide $x^\ell(x^{k-\ell} \pm 1)$. Hence, $T_n(f_1(x))$ and $T_n(f_2(x))$ are disjoint.

For each $f(x) \in S_n$ divisible by $g(x)^2$, there correspond n polynomials, namely the elements of $T_n(f(x))$, which are not divisible by $g(x)^2$, and these n polynomials are different for different $f(x)$. Thus, there are $\leq 2^n/(n+1)$ polynomials in S_n divisible by $g(x)^2$. Hence, there are $o(2^n)$ polynomials in S_n divisible by $g(x)^2$ and thus $o(2^n)$ polynomials $f(x) \in S_n$ which are divisible by the square of a polynomial of degree $\leq R$.

Fix $\varepsilon > 0$. It suffices to show that if R is sufficiently large, then there are $\leq \varepsilon 2^n$ polynomials $f(x) \in S_n$ which are divisible by the square of a polynomial in $\mathbb{Z}[x]$ of degree $> R$. We will use Theorem 1 with $b = 4$ and the fact already established that the roots of the polynomials in S_n have absolute value < 2. We note, however, the case $b = 3$ of Theorem 1 could be used instead of the case $b = 4$ if we use that the roots of the polynomials in S_n have real parts < 1.5 (cf. [1] or [2]).

Let $f(x) \in S_n$ with $f(x)$ divisible by the square of a polynomial $g(x) \in \mathbb{Z}[x]$ of degree $r > R$. We may suppose that $g(x)$ is monic (otherwise, replace $g(x)$ with $-g(x)$). Then the roots of $f(x)$ and hence $g(x)$ have absolute value < 2. If $\beta_1, \dots, \beta_r$ denote the roots of $g(x)$, then $g(x) = \prod_{j=1}^r (x - \beta_j)$ and

$$|g(4)| = \prod_{j=1}^{r} |4 - \beta_j| \geq 2^r > 2^R.$$

Since $f(x)$ is divisible by $g(x)^2$, we deduce that $f(4)$ is divisible by d^2 for some integer $d > 2^R$. On the other hand, from Lemma 9 with $b = 4$, we obtain that for R sufficiently large, there are $\leq \varepsilon 2^n$ such polynomials $f(x) \in S_n$. Hence, Theorem 2 follows.

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