

# ON THE IRREDUCIBILITY OF THE GENERALIZED LAGUERRE POLYNOMIALS

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## 1 Introduction

The generalized Laguerre polynomials are defined by

$$L_m^{(\alpha)}(x) = \sum_{j=0}^m \frac{(m+\alpha)(m-1+\alpha)\cdots(j+1+\alpha)(-x)^j}{(m-j)!j!},$$

where  $m$  is a positive integer and  $\alpha$  is an arbitrary complex number. In 1929, I. Schur [4] established the irreducibility over the rationals of  $L_m^{(0)}(x)$ , the classical Laguerre polynomials, for every  $m$ . In 1931, I. Schur [5] considered  $L_m^{(\alpha)}(x)$  in general and showed that  $L_m^{(1)}(x)$  is irreducible over the rationals for every  $m$ . The case  $\alpha \notin \{0, 1\}$  remained open. The purpose of this paper is to establish the following:

**Theorem 1.** *Let  $\alpha$  be a rational number which is not a negative integer. Then for all but finitely many positive integers  $m$ , the polynomial  $L_m^{(\alpha)}(x)$  is irreducible over the rationals.*

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Before going to the proof, it is worth noting that reducible  $L_m^{(\alpha)}(x)$  do exist even with  $\alpha = 2$ . In particular, we give the following examples:

$$L_2^{(2)}(x) = \frac{1}{2}(x-2)(x-6)$$

$$L_2^{(23)}(x) = \frac{1}{2}(x-20)(x-30)$$

$$L_4^{(23)}(x) = \frac{1}{24}(x-30)(x^3 - 78x^2 + 1872x - 14040)$$

$$L_4^{(12/5)}(x) = \frac{1}{15000}(25x^2 - 420x + 1224)(25x^2 - 220x + 264)$$

$$L_5^{(39/5)}(x) = \frac{-1}{375000}(5x-84)(625x^4 - 29500x^3 + 448400x^2 - 2662080x + 5233536).$$

It is not difficult to show that in fact there are infinitely many positive integers  $\alpha$  for which  $L_2^{(\alpha)}(x)$  is reducible (a product of two linear polynomials).

Theorem 1 is a direct consequence of the following more general result:

**Theorem 2.** *Let  $\alpha$  be a rational number which is not a negative integer. Then for all but finitely many positive integers  $m$ , the polynomial*

$$\sum_{j=0}^m a_j \frac{(m+\alpha)(m-1+\alpha)\cdots(j+1+\alpha)x^j}{(m-j)!j!}$$

*is irreducible over the rationals provided only that  $a_j \in \mathbb{Z}$  for  $0 \leq j \leq m$  and  $|a_0| = |a_m| = 1$ .*

I. Schur obtained his irreducibility results for  $L_m^{(0)}(x)$  and  $L_m^{(1)}(x)$  through general results similar to the above. Recent work of a similar nature has been done by Filaseta [1, 2] and by Filaseta and Trifonov [3]. We note also that the above results can be made effective so that for any fixed  $\alpha \in \mathbb{Q}$ ,  $\alpha$  not a negative integer, it is possible to determine a finite set  $S = S(\alpha)$  of  $m$  such that the polynomial in Theorem 2 is irreducible (for  $a_j$  as stated there) provided  $m \notin S$ .

## 2 A Proof of Theorem 2

For a prime  $p$  and a non-zero integer  $a$ , we define  $\nu(a) = \nu_p(a) = e$  where  $p^e || a$ . We set  $\nu(0) = +\infty$ . We begin with the following preliminary results.

**Lemma 1.** *Let  $k$  be a positive integer. Suppose  $g(x) = \sum_{j=0}^n b_j x^j \in \mathbb{Z}[x]$  and  $p$  is a prime such that  $p \nmid b_n$ ,  $p \mid b_j$  for all  $j \in \{0, 1, \dots, n-k\}$ , and  $\nu(b_j) > \nu(b_0) - (j/k)$  for  $1 \leq j \leq n$ . Then for any integers  $a_0, a_1, \dots, a_n$  with  $|a_0| = |a_n| = 1$ , the polynomial  $f(x) = \sum_{j=0}^n a_j b_j x^j$  cannot have a factor of degree  $k$  in  $\mathbb{Z}[x]$ .*

**Lemma 2.** *Let  $a, b, c$  and  $d$  be integers with  $bc - ad \neq 0$ . Then the largest prime factor of  $(am + b)(cm + d)$  tends to infinity as the integer  $m$  tends to infinity.*

Lemma 1 is a consequence of Lemma 2 in [1]. Observe that  $f(x)$  satisfies the same conditions as  $g(x)$  in the lemma so that the lemma can be established by simply showing the conditions on  $g(x)$  imply  $g(x)$  cannot have a factor of degree  $k$  (see [1] for details). Lemma 2 above is a fairly easy consequence of the fact that the Thué equation  $ux^3 - vy^3 = w$  has finitely many solutions in integers  $x$  and  $y$  where  $u, v$ , and  $w$  are fixed integers with  $w \neq 0$ . It also immediately follows from Corollary 1.2 of [6]. We omit the proofs.

Fix  $\alpha$  now as in Theorem 2. Throughout the argument we suppose as we may that  $m$  is large. Define

$$c_j = \binom{m}{j} (m + \alpha)(m - 1 + \alpha) \cdots (j + 1 + \alpha) \quad \text{for } 0 \leq j \leq m.$$

We want to show that for all but finitely many positive integers  $m$ , the polynomial  $f(x) = \sum_{j=0}^m a_j c_j x^j$  is irreducible over the rationals, where  $a_j$  are arbitrary integers with  $|a_0| = |a_m| = 1$ . Motivated by Lemma 1, we consider instead  $g(x) = \sum_{j=0}^m c_j x^j$ . Let  $u$  and  $v$  be relatively prime integers with  $v > 0$  such that  $\alpha = u/v$ . The condition that  $\alpha$  is not a negative integer implies that for each  $j \in \{0, 1, \dots, m-1\}$ ,  $m - j + \alpha$  and, hence,  $v(m - j) + u$  cannot be zero. We assume that  $g(x)$  has a factor in  $\mathbb{Z}[x]$  of degree  $k \in [1, m/2]$  and establish the theorem by obtaining a contradiction to Lemma 1. We divide the argument into cases depending on the size of  $k$ .

*Case 1.*  $k > m/\log^2 m$ .

For  $a$  and  $b$  integers with  $b > 0$ , let  $\pi(x; b, a)$  denote the number of primes  $\leq x$  which are  $\equiv a \pmod{b}$ . Then the Prime Number Theorem for Arithmetic Progressions implies that if  $\gcd(a, b) = 1$ , then

$$\begin{aligned} \pi(x; b, a) &= \frac{1}{\phi(b)} \int_2^x \frac{dt}{\log t} + O\left(\frac{x}{\log^4 x}\right) \\ &= \frac{1}{\phi(b)} \left( \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + O\left(\frac{x}{\log^4 x}\right) \right). \end{aligned}$$

By considering  $\pi(x; b, a) - \pi(x - h; b, a)$ , it follows that for  $a$  and  $b$  fixed, the interval  $(x - h, x]$  contains a prime  $\equiv a \pmod{b}$  if  $h = x/(2 \log^2 x)$  and if  $x$  is sufficiently large. Taking  $a = u$ ,  $b = v$ , and  $x = vm + u$ , we deduce that for some integer  $j \in [0, k)$ , the number  $v(m - j) + u$  is prime. Call such a prime  $p$ , and observe that  $p \geq 2vm/3$  (since  $v$  is a positive integer and  $m$  is large). We deduce that  $p$  does not divide  $v$ . Observe that

$$c_\ell = \binom{m}{\ell} \frac{(vm + u)(v(m - 1) + u) \cdots (v(\ell + 1) + u)}{v^{m-\ell}} \quad \text{for } 0 \leq \ell \leq m.$$

For  $j \in \{0, 1, \dots, k - 1\}$ , the numbers  $v(m - j) + u$  appear in the numerator of the fraction on the right-hand side above whenever  $0 \leq \ell \leq m - k$ . Therefore,

$$(1) \quad \nu_p(c_\ell) \geq 1 \quad \text{for } 0 \leq \ell \leq m - k.$$

Since  $c_m = 1$ ,  $\nu_p(c_m) = 0$ . To obtain a contradiction from Lemma 1 for the case under consideration, we show that  $\nu_p(c_0) = 1$ ; the contradiction will be achieved since (1) and  $k \leq m - k$  imply  $\nu(c_\ell) \geq 1 > 1 - (\ell/k)$  for  $1 \leq \ell \leq k$  and since the inequality  $\nu(c_\ell) > 1 - (\ell/k)$  is clear for  $k < \ell \leq m$ . Recall that  $p \nmid v$  and that  $p \geq 2vm/3$ . For  $j \in \{0, 1, \dots, m - 1\}$ , we deduce the inequality

$$2p > vm + u \geq v(m - j) + u \geq v + u > -p.$$

The condition that  $\alpha$  is not a negative integer implies that none of  $v(m - j) + u$  can be zero. Hence,  $p$  itself is the only multiple of  $p$  among the numbers  $v(m - j) + u$  with  $0 \leq j \leq m - 1$ . Since  $c_0 = (vm + u)(v(m - 1) + u) \cdots (v + u)/v^m$ , we obtain  $\nu_p(c_0) = 1$ .

*Case 2.*  $k_0 \leq k \leq m/\log^2 m$  with  $k_0 = k_0(u, v)$  a sufficiently large integer.

Let  $z = k(\log \log k)^{1/2}$ . We first show that there is a prime  $p > z$  that divides  $v(m - j) + u$  for some  $j \in \{0, 1, \dots, k - 1\}$ . Then (1) follows as before, and we will obtain a contradiction to Lemma 1 by showing  $\nu(c_j) > \nu(c_0) - (j/k)$  for  $1 \leq j \leq m$ .

Let

$$T = \{v(m - j) + u : 0 \leq j \leq k - 1\}.$$

Since  $m$  is large, we deduce that the elements of  $T$  are each  $\geq m/2$ . Also, observe that  $\gcd(u, v) = 1$  implies that each element of  $T$  is relatively prime to  $v$ . For each prime  $p \leq z$ , we consider an element  $a_p = v(m - j) + u \in T$  with  $\nu_p(a_p)$  as large as possible. We let

$$S = T - \{a_p : p \nmid v, p \leq z\}.$$

By the Prime Number Theorem,

$$\pi(z) \leq \frac{2k(\log \log k)^{1/2}}{\log k}.$$

We combine this momentarily with  $|S| \geq k - \pi(z)$ . Since  $k \leq m/\log^2 m$ , we obtain  $m \geq k \log^2 k$ . Consider a prime  $p \leq z$  with  $p$  not dividing  $v$ , and let  $r = \nu_p(a_p)$ . By the definition of  $a_p$ , if  $j > r$ , then there are no multiples of  $p^j$  in  $T$  (and, hence, in  $S$ ). For  $1 \leq j \leq r$ , there are  $\leq [k/p^j] + 1$  multiples of  $p^j$  in  $T$  and, hence, at most  $[k/p^j]$  multiples of  $p^j$  in  $S$ . Therefore,

$$\nu_p \left( \prod_{s \in S} s \right) \leq \sum_{j=1}^r \left[ \frac{k}{p^j} \right] \leq \nu_p(k!),$$

and

$$\prod_{s \in S} \prod_{p \leq z} p^{\nu_p(s)} \leq k! \leq k^k.$$

On the other hand,

$$\prod_{s \in S} s \geq \left( \frac{m}{2} \right)^{|S|} \geq \left( \frac{k \log^2 k}{2} \right)^{k - \pi(z)}.$$

Recalling our bound on  $\pi(z)$ , we obtain

$$\begin{aligned} \log \left( \prod_{s \in S} s \right) &\geq (k - \pi(z))(\log k + 2 \log \log k - \log 2) \\ &\geq \left( k - \frac{2k\sqrt{\log \log k}}{\log k} \right) (\log k + 2 \log \log k - \log 2) \\ &\geq k \log k + 2k \log \log k + O(k\sqrt{\log \log k}). \end{aligned}$$

Since  $k \geq k_0$  where  $k_0$  is sufficiently large,

$$\log \left( \prod_{s \in S} s \right) > k \log k \geq \log \left( \prod_{s \in S} \prod_{p \leq z} p^{\nu_p(s)} \right).$$

It follows that there is a prime  $p > z$  that divides some element of  $S$  and, hence, divides some element of  $T$ .

Fix a prime  $p > z$  that divides an element in  $T$ , and let  $\nu = \nu_p$ . Fix  $j \in \{1, 2, \dots, m\}$ . We show  $\nu(c_j) > \nu(c_0) - (j/k)$ . Observe that

$$\begin{aligned} \nu(c_0) - \nu(c_j) &\leq \nu((vj + u)(v(j-1) + u) \cdots (v + u)) \\ &\leq \nu((vj + |u|)!) = \sum_{j=1}^{\infty} \left\lfloor \frac{vj + |u|}{p^j} \right\rfloor \\ &< \sum_{j=1}^{\infty} \frac{vj + |u|}{p^j} = \frac{vj + |u|}{p-1}. \end{aligned}$$

Since  $p > z = k(\log \log k)^{1/2}$  and  $k \geq k_0$ , we deduce that  $(vj + |u|)/(p-1) < j/k$  and the inequality  $\nu(c_j) > \nu(c_0) - (j/k)$  follows. Hence, as indicated at the beginning of this case, we obtain a contradiction to Lemma 1.

*Case 3.*  $2 \leq k < k_0$ .

By Lemma 2 (with  $a = v$ ,  $b = u$ ,  $c = v$ , and  $d = u - v$ ), the largest prime factor of the product  $(vm+u)(v(m-1)+u)$  tends to infinity. Since  $m$  is large, we deduce that there is a prime  $p > (v + |u|)k_0$  that divides  $(vm+u)(v(m-1)+u)$ . The argument now follows as in the previous case. In particular,

$$\frac{\nu(c_0) - \nu(c_j)}{j} < \frac{vj + |u|}{j(p-1)} \leq \frac{v + |u|}{p-1} \leq \frac{1}{k_0} < \frac{1}{k} \quad \text{for } 1 \leq j \leq m.$$

Hence, in this case, we also obtain a contradiction.

*Case 4.*  $k = 1$ .

From Lemma 2, the largest prime factor of  $m(vm+u)$  tends to infinity with  $m$ . We consider a large prime factor  $p$  of this product. In particular, we suppose that  $p > v + |u|$ . Note this implies  $p \nmid v$ . As in the previous case, we are through if  $p \mid (vm+u)$ . So suppose  $p \mid m$ . The binomial coefficient  $\binom{m}{j}$  appears in the definition of  $c_j$ , and this is sufficient to guarantee that  $\nu(c_j) \geq 1$  and  $\nu(c_{m-j}) \geq 1$  for  $1 \leq j \leq p-1$ . On the other hand,

$$c_j = \binom{m}{j} \frac{(vm+u)(v(m-1)+u) \cdots (v(j+1)+u)}{v^{m-j}}.$$

For  $j \leq m-p$ , the numerator of the fraction on the right is a product of  $\geq p$  consecutive terms in the arithmetic progression  $vt + u$  with  $\gcd(p, v) = 1$ ; thus,  $\nu(c_{m-j}) \geq 1$  for  $j \geq p$ . This implies that (1) holds with  $k = 1$ . It follows, along the lines of the previous two cases, that  $\nu(c_j) > \nu(c_0) - (j/k)$  for  $1 \leq j \leq m$ . A contradiction to Lemma 1 is again obtained (and the proof of the theorem is complete).

## References

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