Newton polygons and
the Prouhet-Tarry-Escott problem

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1 Introduction

For $n \geq 2$, we consider two lists of integers

$$X = [x_1, x_2, \ldots, x_n] \quad \text{and} \quad Y = [y_1, y_2, \ldots, y_n],$$

where, for this section only, we view these as ordered so that $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. We also require $x_j \neq y_j$ for at least one $j \in \{1, 2, \ldots, n\}$. The Prouhet-Tarry-Escott problem (the PTE problem) asks for such X and Y satisfying

$$\sum_{i=1}^{n} x_i^e = \sum_{i=1}^{n} y_i^e \qquad \text{for } e = \{1, 2, \ldots, k\} \tag{1}$$

where k is an integer in the interval $[2, n-1]$. If X and Y satisfy (1) then the pair is called a *s*olution of the PTE problem, denoted as $X =_k Y$. A solution is *ideal* if $k = n - 1$. The significance of the case $k = n - 1$ is that with X and Y distinct as required above, it is impossible for (1) to hold if $k \geq n$. Thus, the largest possible value for k in (1) is $n - 1$.

Literature on the PTE problem is extensive. The problem is a focus of an entire chapter (Chapter 24) of L. E. Dickson's classical volumes "History of the Theory of Numbers" [9] and numerous early references can be found there. The problem is also discussed in G. H. Hardy and E. M. Wright's well-known "An Introduction to the Theory of Numbers" [12], undoubtedly in part due to Wright's own interest in the problem (cf. [21, 22, 23]). We note that for the first half of the twentieth century, the problem was referred to as the Tarry-Escott problem, until Wright [22] pointed out that E. Prouhet [17] first discussed the problem in 1851. A few of the more recent investigations on the PTE problem include [4, 5, 8, 14, 18]. Interesting work on generalizations of the PTE problem can be found in [1, 6]. For applications arising from the PTE problem see [2, 11, 13, 16, 19].

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An important open problem in the area is a conjecture of Wright [21] that for every natural number $n \geq 3$, an ideal solution exists. Despite its long history, ideal solutions are only known to exist for $3 \leq n \leq 10$ and $n = 12$. In particular, no ideal solution is known for $n = 11$.

To help formulate further discussion, we note that the following result and its corollary are fairly simple consequences of properties of elementary symmetric functions (see [3, 4]).

Lemma 1. *Let n and k be integers with $1 \leq k < n$. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ denote arbitrary integers. The following are equivalent:*

- $\displaystyle\sum_{i=1}^{n} x_i^e = \sum_{i=1}^{n} y_i^e, \quad$ *for* $e \in \{1, 2, \ldots, k\}$,
- $\displaystyle\deg\left(\prod_{i=1}^{n}(z - x_i) - \prod_{i=1}^{n}(z - y_i)\right) \leq n - k - 1$,
- $\displaystyle(z-1)^{k+1} \Big| \left(\sum_{i=1}^{n} z^{x_i} - \sum_{i=1}^{n} z^{y_i}\right)$.

Corollary 1. *The lists $X = [x_1, \ldots, x_n]$ and $Y = [y_1, \ldots, y_n]$ give an ideal PTE solution if and only if*

$$\prod_{i=1}^{n}(z - x_i) - \prod_{i=1}^{n}(z - y_i) = C \tag{2}$$

for some real constant C.

In this paper, we will view ideal PTE solutions as being lists X and Y satisfying (2). For computational reasons (see [4, 7, 18]), information on possible values of C and, in particular, on the factorization of C given (2), has played an important role in arriving at examples of ideal PTE solutions. As C depends on n, X and Y, we define, for $X =_{n-1} Y$, the constant

$$C_n = C_n(X, Y) = \prod_{i=1}^{n}(z - x_i) - \prod_{i=1}^{n}(z - y_i).$$

We clarify that what is of interest here then is the value of

$$\overline{C}_n = \prod_{j=1}^{\infty} p_j^{e_j},$$

where

$$e_j = \min\{e : p_j^e \| C_n(X, Y) \text{ for some } X \text{ and } Y \text{ as above with } X =_{n-1} Y\}.$$

In other words, $\overline{C}_n$ can be viewed as the greatest common divisor over all constants $C_n(X, Y)$ where X and Y vary over distinct ordered lists of n integers satisfying $X =_{n-1} Y$. So we would like to know, for a given n, how $\overline{C}_n$ factors.

With the notation above, we state the following result that plays a role throughout the paper; it is an easy consequence of Corollary 1 or Lemma 1.

Corollary 2. *Let $a \in \mathbb{Z}$. The pair of lists $X = [x_1, \ldots, x_n]$ and $Y = [y_1, \ldots, y_n]$ form an ideal PTE solution if and only if the pair of lists $X' = [x_1 + a, \ldots, x_n + a]$ and $Y' = [y_1 + a, \ldots, y_n + a]$ form an ideal PTE solution. Furthermore, if these are ideal solutions, then $C_n(X, Y) = C_n(X', Y')$.*

The values of $\overline{C}_n$ for $3 \leq n \leq 7$ are known (see [7]):

$$\overline{C}_3 = 2^2$$

$$\overline{C}_4 = 2^2 \cdot 3^2$$

$$\overline{C}_5 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$$

$$\overline{C}_6 = 2^5 \cdot 3^2 \cdot 5^2$$

$$\overline{C}_7 = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11.$$

In this paper, we pay particular attention to ideal solutions of sizes 8 and 9. For these, according to [7], it is known that

$$\overline{C}_8 = 2^{e_1} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13, \quad \text{where } 4 \leq e_1 \leq 8$$

$$\overline{C}_9 = 2^{e_2} \cdot 3^{e_3} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^{e_4} \cdot 23^{e_5} \cdot 29^{e_6}, \quad \text{where } 7 \leq e_2 \leq 9,\ 3 \leq e_3 \leq 4,\ 0 \leq e_j \leq 1, \text{ for } j \in \{4, 5, 6\}.$$

There are two noteworthy examples that pertain to this paper. L. E. Dickson [9] reports that, in 1913, G. Tarry [20] observed that

$$(x^2 - 5^2)(x^2 - 14^2)(x^2 - 23^2)(x^2 - 24^2) - (x^2 - 2^2)(x^2 - 16^2)(x^2 - 21^2)(x^2 - 25^2) = C$$

where $C = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$. According to E. M. Wright [23], in 1942, A. Létac [15] gave the example

$$\begin{aligned} &(z-1)(z-25)(z-31)(z-84)(z-87)(z-134)(z-158)(z-182)(z-198) \\ &\quad -(z-2)(z-18)(z-42)(z-66)(z-113)(z-116)(z-169)(z-175)(z-199) \\ &\quad = 3377425033382400 = 2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \cdot 41. \end{aligned}$$

These imply

$$\nu_2\big(\overline{C}_8\big) \leq 8 \quad \text{and} \quad \nu_2\big(\overline{C}_9\big) \leq 9,$$

where $\nu_2(m)$ refers to the 2-adic value of m, that is the largest integer j for which $2^j \| m$. These somewhat old examples then give the upper bounds described above for e_1 and e_2.

Our interest in this paper is to explain how the classical theory of Newton polygons can be used to obtain information about the 2-adic values of $\overline{C}_n$. In particular, we show that $\nu_2\big(\overline{C}_9\big) = 9$. For $n = 8$, we only provide the inequality $\nu_2\big(\overline{C}_8\big) \geq 6$.

The arguments we give take advantage of working modulo small powers of 2 and substituting values for z in (2) to show smaller values for $\nu_2\big(\overline{C}_9\big)$ and $\nu_2\big(\overline{C}_8\big)$ cannot exist. We have found the example

$$X = [31914804930538, 392011859134314, 414199788923609,$$

$$550721232905543, 563570240533272, 870589495146520,$$
$$1039460985683225, 1113937730497799]$$

and

$$Y = [226375709153429, 382003430459158, 502458387218286,$$
$$690280771238587, 750383096702563, 764464731978500,$$
$$790357673966989, 870082337037308]$$

which has the property that

$$\prod_{i=1}^{8}(z - x_i) - \prod_{i=1}^{8}(z - y_i) \equiv 954668492881984 \pmod{2^{50}}.$$

Of interest here is that the number 954668492881984 is exactly divisible by 2^6. Thus, there is no real hope that working modulo small powers of 2 will enable one to show 2^7 must divide C in (2). Further, substituting any $z \in \mathbb{Z}$ into the expression on the left above results in an integer exactly divisible by 2^6, so such substitutions will not provide us with a means to show 2^7 must divide C. Perhaps examples like the above exist for the obvious reason that $2^6 \| \overline{C}_8$, and an appropriate example, different from the one of Tarry's indicated above, is needed then to show that $\nu_2(\overline{C}_8) = 6$.

The example above raises some natural questions. Is it possible to show that a 2-adic ideal solution exists for the PTE problem for every $n \geq 3$? Let p be a prime. Does a p-adic ideal solution necessarily exist for $n = 11$? Is it possible to have a p-adic solution to

$$\prod_{i=1}^{n}(z - x_i) - \prod_{i=1}^{n}(z - y_i) = C,$$

for which $\nu_p(C) < \nu_p(\overline{C}_n)$, where ν_p is the usual p-adic valuation and n is some integer ≥ 3?

2 Further preliminaries

We write

$$f(z) = \prod_{j=1}^{n}(z - x_j) = \sum_{j=0}^{n} a_j z^j \quad \text{and} \quad g(z) = \prod_{j=1}^{n}(z - y_j) = \sum_{j=0}^{n} b_j z^j$$

where $x_j, y_j \in \mathbb{Z}$ are chosen so that

$$f(z) - g(z) = C_n \tag{3}$$

and so that the exact power of 2 dividing C_n is equal to the exact power of 2 dividing $\overline{C}_n$. Thus, by Corollary 1, we have that $X = [x_1, \ldots, x_n]$ and $Y = [y_1, \ldots, y_n]$ is an ideal solution. We write $C = C_n$, where n should be clear from the context.

For fixed n, we consider the two sets of points in the extended plane

$$S_1 = \{(j, \nu_2(a_{n-j})) : 0 \le j \le n\} \quad \text{and} \quad S_2 = \{(j, \nu_2(b_{n-j})) : 0 \le j \le n\}.$$

Since $f(z) - g(z) = C$, a constant, we see that $a_{n-j} = b_{n-j}$ for $0 \le j \le n-1$. Thus, S_1 and S_2 have at least n of $n+1$ points in common.

Recalling Corollary 2, we translate $f(z)$ and $g(z)$ by the same translation, if necessary, so that $a_0 \ne 0$ and $b_0 \ne 0$. Thus, $\nu_2(a_0) \ne +\infty$ and $\nu_2(b_0) \ne +\infty$. Note that (3) still holds. This ensures that the right-most points $(n, \nu_2(a_0))$ and $(n, \nu_2(b_0))$, which may differ in S_1 and S_2, are in the finite plane.

We will be interested in Newton polygons, and in particular to a result that goes back to work of G. Dumas [10].

Definition 1. *Let $F(z) = \sum_{j=0}^{n} c_j z^j \in \mathbb{Z}[z]$ with $c_0 c_n \ne 0$. Let p be a prime. For $j \in \{0, \cdots, n\}$, we define $x_j = j$ and define $y_j = \nu_p(c_{n-j})$. We consider the lower edges along the convex hull of the points in $S = \{(x_0, y_0), \cdots, (x_n, y_n)\}$. The polygonal path formed by these edges is called the Newton polygon associated with $F(z)$ with respect to p.*

Thus, the Newton polygon of $f(z)$ with respect to the prime 2 is the lower convex hull of the points in S_1, and the Newton polygon of $g(z)$ with respect to 2 is the lower convex hull of the points in S_2. Note that the slopes of the edges of the Newton polygons increase from left to right. We state next an important property of Newton polygons based on the set-up in this paper.

Lemma 2. *The Newton polygons of $f(z)$ and $g(z)$ will each pass through $n+1$ lattice points (including the endpoints), which we denote respectively as*

$$T_1 = \{(j, t_j) : 0 \le j \le n\} \quad \textit{and} \quad T_2 = \{(j, t'_j) : 0 \le j \le n\}.$$

After possibly rearranging the x_j and y_j, we have $2^{t_j - t_{j-1}}$ exactly divides x_j and $2^{t'_j - t'_{j-1}}$ exactly divides y_j for each $j \in \{1, 2, \cdots, n\}$.

This lemma follows directly from a theorem of Dumas [10] which asserts that the Newton polygon of a product of two polynomials with respect to a prime p can be obtained by translating the edges of the Newton polygons for each polynomial with respect to p. Since $f(z)$ and $g(z)$ are a product of n linear factors, we have that the Newton polygons associated with $f(z)$ and $g(z)$ each consists of n line segments translated so that $n+1$ lattice points (including endpoints) are along its edges. Each translated segment will have the x-coordinates of its endpoints differing by 1.

As a consequence of Lemma 2, the slope of each edge of the Newton polygon of $f(z)$ and $g(z)$ is an integer. In the last statement of Lemma 2, observe that the values $\nu_2(x_j)$ and $\nu_2(y_j)$ are increasing as j ranges from 1 to n. We will want to use such an ordering throughout the remainder of the paper. In particular, the values of the x_j and the values of the y_j themselves are not necessarily increasing as in the introduction.

To illustrate, we consider $n = 9$ and take the example of A. Létac [15] mentioned in the introduction, so

$$X = [1, 25, 31, 87, 134, 158, 182, 198, 84] \quad \text{and} \quad Y = [113, 169, 175, 199, 2, 18, 42, 66, 116],$$

where we have taken an ordering of the x_j and y_j corresponding to the last statement in Lemma 2. In this case,

$$f(z) \equiv g(z) \equiv z^9 + 124z^8 + 70z^7 + 24z^6 + 33z^5 + 12z^4 + 72z^3 + 32z^2 + 80z + 64 \pmod{128},$$

so that the Newton polygons of $f(z)$ and of $g(z)$ with respect to 2 look the same and are as shown in Figure 1.

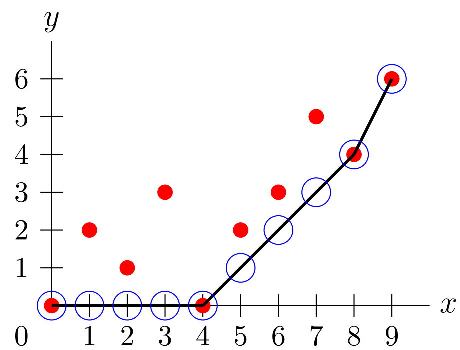

Figure 1: Newton polygon for A. Létac's example

The solid circles represent the points of S_1 and S_2 with the bottom left-hand endpoint equal to $(0, 0)$ in each case (since the polynomials are monic). The open circles refer to the lattice points in T_1 and T_2 as mentioned in Lemma 2. Thus, for this example,

$$T_1 = T_2 = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 1), (6, 2), (7, 3), (8, 4), (9, 6)\}.$$

As implied by Lemma 2, the height differences between two consecutive lattice points in T_1 indicates that there are exactly four odd x_j's, four x_j's that are exactly divisible by 2, and one x_j exactly divisible by 4. As $T_1 = T_2$, the y_j's satisfy analogous conditions. We note that despite this example, in general, unlike S_1 and S_2 which have all but possibly their right-most points in common, the points other than $(0, 0)$ belonging to T_1 and T_2 can be different.

Lemma 3. *If the points* $(n, \nu_2(a_0))$ *in* S_1 *and* $(n, \nu_2(b_0))$ *in* S_2 *are distinct and*

$$k = \min\{\nu_2(a_0), \nu_2(b_0)\},$$

then $2^k \| C$.

Proof. Since $C = a_0 - b_0$ and $\nu_2(a_0) \neq \nu_2(b_0)$, we see that

$$\nu_2(C) = \nu_2(a_0 - b_0) = \min\{\nu_2(a_0), \nu_2(b_0)\} = k.$$

Thus, $2^k \| C$. □

We develop some notation that we will be using in the subsequent sections. Let k_1 be the number of odd x_j and k_1' be the number of odd y_j; thus, the 2-valuation of each of these x_j and y_j is equal to 0. Further, we let k_2 be the number of x_j which are congruent to $2 \pmod 4$ and k_2' be the number of y_j that are congruent to $2 \pmod 4$; thus, the 2-valuation of each of these x_j and y_j is equal to 1.

By translating $f(z)$ and $g(z)$ by 1 (or some odd number to guarantee that a_0 and b_0 are not equal to 0), we may suppose $k_1' \leq \lfloor n/2 \rfloor$. Furthermore, we may now translate by 2 (or some other number that is congruent to $2 \pmod 4$) if needed to obtain that $k_2' \geq \lceil (n-k_1')/2 \rceil$ of the y_j are congruent to $2 \pmod 4$.

Using the following proposition from [4], we deduce that if C is even, then $k_1 = k_1'$.

Lemma 4. *Let $[x_1, \ldots, x_n] =_{n-1} [y_1, \ldots, y_n]$ be two lists of integers that constitute an ideal PTE solution, and suppose that a prime p divides the constant C associated with this solution. Then we can reorder the integers y_i so that*

$$x_j \equiv y_j \pmod p \qquad \text{for } j \in \{1, \ldots, n\}.$$

As noted, we can deduce now that the number of odd x_j must equal the number of odd y_j, that is, $k_1 = k_1'$. Further, we can interchange the roles of $f(z)$ and $g(z)$, if necessary, so that $k_2' \geq k_2$. Since there are n elements in the lists X and Y, it must be the case that $k_1 + k_2 \leq n$ and $k_1' + k_2' \leq n$.

Before ending this section, we establish the following.

Lemma 5. *Let $n \geq 8$. Suppose $[x_1, \ldots, x_n] =_{n-1} [y_1, \ldots, y_n]$. For $1 \leq j \leq n$, let x_j and y_j be such that $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ are odd and otherwise x_j and y_j are even. Then*

$$x_1^k + \cdots + x_t^k \equiv y_1^k + \cdots + y_t^k \pmod{16}, \quad \text{for } k \geq 1.$$

and

$$x_{t+1}^k + \cdots + x_n^k \equiv y_{t+1}^k + \cdots + y_n^k \pmod{16}, \quad \text{for } k \geq 1. \tag{4}$$

Proof. Since $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ are odd, we obtain

$$x_j^4 \equiv y_j^4 \equiv 1 \pmod{16}, \quad \text{for } 1 \leq j \leq t.$$

Thus,

$$x_1^k + \cdots + x_t^k \equiv x_1^{k+4} + \cdots + x_t^{k+4} \pmod{16}$$

and

$$y_1^k + \cdots + y_t^k \equiv y_1^{k+4} + \cdots + y_t^{k+4} \pmod{16}.$$

As $x_j^{k+4} \equiv y_j^{k+4} \equiv 0 \pmod{16}$ for $t+1 \leq j \leq n$, we deduce that

$$\begin{aligned} x_1^k + \cdots + x_t^k &\equiv x_1^{k+4} + \cdots + x_t^{k+4} \equiv x_1^{k+4} + \cdots + x_n^{k+4} \\ &\equiv y_1^{k+4} + \cdots + y_n^{k+4} \equiv y_1^{k+4} + \cdots + y_t^{k+4} \equiv y_1^k + \cdots + y_t^k \pmod{16}, \end{aligned}$$

provided $1 \leq k+4 \leq n-1$. Since $n \geq 8$, the above holds for $1 \leq k \leq 3$. On the other hand,

$$x_1^k + \cdots + x_n^k = y_1^k + \cdots + y_n^k \qquad \text{for } 1 \leq k \leq 3.$$

Hence,

$$x_{t+1}^k + \cdots + x_n^k \equiv y_{t+1}^k + \cdots + y_n^k \pmod{16} \qquad \text{for } 1 \leq k \leq 3.$$

The lemma follows since for $k \geq 4$, both sides of the congruence in (4) are divisible by 16. □

Corollary 3. *Let* $n \geq 8$. *Suppose* $[x_1, \ldots, x_n] =_{n-1} [y_1, \ldots, y_n]$. *Let* k_1, k_1', k_2 *and* k_2' *be as above. Then* $k_2 \equiv k_2' \pmod 4$.

Proof. Recall $k_1 = k_1'$. From Lemma 5, we have

$$x_{k_1+1}^2 + x_{k_1+1}^2 + \cdots + x_n^2 \equiv x_{k_1+1}^2 + x_{k_1+1}^2 + \cdots + x_n^2 \pmod{16}.$$

As an even integer m squared is 4 modulo 16 if $m \equiv 2 \pmod 4$ and otherwise is 0 modulo 16, the above congruence can be rewritten as $4k_2 \equiv 4k_2' \pmod{16}$. The result follows. □

3 The 2-adic value of $\overline{C}_9$

Recall that it is known that $2^7|\overline{C}_9$ and $2^{10} \nmid \overline{C}_9$. Our goal in this section is to increase the lower bound of the valuation of 2 in $\overline{C}_9$. With the aid of Newton polygons, we establish $2^9|\overline{C}_9$ from which we can deduce that $2^9\|\overline{C}_9$.

We make use of the notation in the previous section with $n = 9$ and deal with two cases, each involving multiple subcases, depending on the values of k_1' and k_2'.

Case 1. $k_1' + k_2' = 9$

In this case, we are assuming that there are no elements in the list Y that are congruent to $0 \pmod 4$. We consider possibilities for the Newton polygon of $f(z)$. From Lemma 4, we know that $k_1 = k_1'$ odd x_j's are in the list X. We recall that $k_2 \leq k_2'$, which implies that X contains at most k_2' elements that are divisible by 2 and not 4. Combining these facts, we have that each point $(j, \nu_2(a_{9-j}))$ in S_1 is on or above the corresponding point $(j, \nu_2(b_{9-j}))$ in S_2.

Case 1.1. $k_2 = k_2'$

Recall that we have translated $f(z)$ and$g(z)$ so that $k_1' \leq \lfloor n/2 \rfloor = \lfloor 9/2 \rfloor = 4$. Therefore, in this subcase, k_2 and k_2' are both greater than or equal to 5. Substituting $z = 2$ in (3), we obtain

$$\prod_{j=1}^{9}(2 - x_j) - \prod_{j=1}^{9}(2 - y_j) = f(2) - g(2) = C,$$

where at least five of the x_j's and at least five of the y_j's are 2 modulo 4. Thus, 2^{10} divides each product, and therefore, their difference. This implies a contradiction, since $2^{10} \nmid C$. In other words, it is impossible for $f(z) - g(z) = C$ with $\nu_2(C) = \nu_2(\overline{C}_9)$ in this case.

Case 1.2. $k_2 < k_2'$

In this subcase, X must contain some elements that are congruent to $0 \pmod 4$ but Y cannot. We deduce that the right-most point of the Newton polygon of $f(z)$ is above the point $(9, \nu_2(b_0))$. Since these endpoints are distinct, by Lemma 3 we have $2^{\nu_2(b_0)}\|C$. Since all of the even elements in Y are congruent to $2 \pmod 4$ (thus have valuation equal to 1 with respect to the prime 2), we have that $\nu_2(b_0) = k_2'$. In the case under consideration, $\nu_2(b_0) = k_2' = 9 - k_1'$. Since we know that $2^7|C$, we have $k_2' \geq 7$ and $k_1' \leq 2$.

Case 1.2.1. $k_1' = 2$

In this case $k_2' = 7$. By Corollary 3, we deduce $k_2 \in \{3, 7\}$. Thus, $2 - x_j$ and $2 - y_j$ are divisible by 4 for $2 \leq j \leq 4$, and $2 - x_j$ and $2 - y_j$ are divisible by 2 for $5 \leq j \leq 8$. Letting $z = 2$ in (3), we see that $2^{10}|C$, giving a contradiction in this case.

Case 1.2.2. $k_1' = 1$

As $k_2' = 8$ in this subcase, Corollary 3 implies $k_2 \in \{0, 4, 8\}$. If $k_2 \geq 4$, then setting $z = 2$ in (3) leads to $2^{12}|C$, giving a contradiction. We are left with considering $k_2 = 0$. Thus, for $j \in \{2, 3, \ldots, 9\}$, we have $4|x_j$.

Observe that setting $z = 2$ in (3) implies $2^8 \| C$. If we now take $z = x_1$ in (3), we obtain

$$C = f(x_1) - g(x_1) = -g(x_1) = -\prod_{j=1}^{9}(x_1 - y_j).$$

As $x_1 - y_j$ is odd for $2 \leq j \leq 9$ and $2^8|C$, we deduce that

$$x_1 \equiv y_1 \pmod{2^8}.$$

Next, we use that $X = [x_1, x_2, \ldots, x_9]$ and $Y = [y_1, y_2, \ldots, y_9]$ being an ideal PTE solution implies

$$x_1^4 + x_2^4 + \cdots + x_9^4 = y_1^4 + y_2^4 + \cdots + y_9^4.$$

Since $x_1 \equiv y_1 \pmod{2^8}$, we easily obtain

$$x_2^4 + x_3^4 + \cdots + x_9^4 \equiv y_2^4 + y_3^4 + \cdots + y_9^4 \pmod{2^8}. \tag{5}$$

For $j \in \{2, 3, \ldots, 9\}$, we can write $y_j = 2(2y_j' + 1)$ for some $y_j' \in \mathbb{Z}$. As $(2y_j' + 1)^4 \equiv 1 \pmod{16}$, we obtain

$$(2y_2' + 1)^4 + (2y_3' + 1)^4 + \cdots + (2y_9' + 1)^4 \equiv 8 \pmod{16},$$

from which it follows that $y_2^4 + y_3^4 + \cdots + y_9^4$ is exactly divisible by 2^7. On the other hand, $4|x_j$ for $j \in \{2, 3, \ldots, 9\}$, so $x_2^4 + x_3^4 + \cdots + x_9^4$ is divisible by 2^8. We obtain a contradiction now from (5), so $f(z) - g(z) = C$ with $\nu_2(C) = \nu_2(\overline{C}_9)$ is impossible in this case.

Case 1.2.3. $k_1' = 0$

From (3),

$$C_9 = f(0) - g(0) = -\prod_{j=1}^{9} x_j + \prod_{j=1}^{9} y_j$$

is divisible by 2^9. This is what we set out to show, so we are done in this case. (Alternatively, one can use that the 18 x_j's and y_j's cannot all have a common prime divisor p in (3) if $\nu_p(C_9)$ is minimal. From this point of view, this subcase cannot occur.)

Case 2. $k_1' + k_2' < 9$

Since $k_1' \leq 4$, we have

$$k_2' \geq \left\lceil \frac{9 - k_1'}{2} \right\rceil \geq 3.$$

We also have $k_2' \geq k_2$. We note the importance of the condition $k_1' + k_2' < 9$. This implies $k_2' < 9 - k_1'$. Hence, $(k_1', 0)$ and $(k_1' + k_2', k_2')$ are points in S_2 with x-coordinates < 9. Therefore, $(k_1', 0)$ and $(k_1' + k_2', k_2')$ are points in S_1. Since there are exactly $k_1 = k_1'$ odd x_j and the Newton polygon of $f(z)$ has integer slopes, we deduce that the segment joining $(k_1', 0)$ and $(k_1' + k_2', k_2')$ is part of the Newton polygon of $f(z)$. In particular, $k_2 \geq k_2' \geq 3$. Since $k_2' \geq k_2$, we deduce $k_2 = k_2' \geq 3$.

Case 2.1. $k_1' \leq 3$

If $k_1' \leq 3$, then there are at least six even x_j and six even y_j. Out of the six even x_j's and the six even y_j's, at least three x_j's and three y_j's are $2 \pmod 4$. Thus, setting $z = 2$ in (3), we obtain $2^9 | C$, as desired.

Case 2.2. $k_1' = 4$

We lastly consider $k_1' = k_1 = 4$ and $k_2 = k_2' \geq 3$. Since we are in the case where $k_1' + k_2' < 9$ and $k_1' = 4$, we have $k_2' < 5$. Thus, either $k_2 = k_2' = 4$ or $k_2 = k_2' = 3$.

Case 2.2.1. $k_2 = k_2' = 4$

If $k_2 = k_2' = 4$, then out of the five even x_j's and the five even y_j's, there are four x_j's and four y_j's that are $2 \pmod 4$. Setting $z = 2$ in (3), we obtain $2^9 | C$ and are done as before.

Case 2.2.2. $k_2 = k_2' = 3$

Recall that the slopes of the Newton polygons of $f(z)$ and $g(z)$ are integers and the slopes increase from left to right. For each of these Newton polygons, the edge with slope 1 ends at the point $(k_1 + k_2, k_2) = (7, 3)$. Thus, the remaining edge(s) to the right have slope at least 2, and therefore, the right-most point on each of the Newton polygons must be on or above $(9, 7)$.

If the right-most points on the Newton polygons, $(9, \nu_2(a_0))$ and $(9, \nu_2(b_0))$, are on or above $(9, 9)$, then we take $z = 0$ in (3) to see that $2^9 | C$. This finishes the argument in this case.

If exactly one of the Newton polygons has the right-most point $(9, 7)$, then we set $z = 2$ in (3) to get $2^8 | C$. However, Lemma 3 implies that $2^7 \| C$, a contradiction.

If both of the Newton polygons have right-most endpoint $(9, 7)$, then by setting $z = 4$ in (3), we see that $2^9 | C$, giving us the conclusion we want.

We now know that one of the Newton polygons has right-most point $(9, 8)$, and the other has right-most endpoint either $(9, 8)$ or above $(9, 8)$. If the right-most endpoint is $(9, 8)$ for one of these Newton polygons, then its two right-most edges consist of the segment joining $(7, 3)$ to $(8, 5)$ and the segment joining $(8, 5)$ to $(9, 8)$. In particular, if $(9, 8)$ is the right-most endpoint for both of the

Newton polygons, then $x_8 \equiv y_8 \equiv 8 \pmod{16}$. Setting $z = 8$ in (3) for this case, we obtain $2^9|C$, as desired.

Finally, we consider the case that one of the Newton polygons has right-most endpoint $(9, 8)$ and the other Newton polygon has right-most endpoint above $(9, 8)$. Recall that the two points $(j, \nu_2(a_{9-j}))$ and $(j, \nu_2(b_{9-j}))$ agree for $j \in \{0, 1, \ldots, 8\}$. We deduce that $(8, 5)$ is a point in either S_1 or S_2, and thus in both. Hence the edge joining $(7, 3)$ and $(8, 5)$ is common to both Newton polygons. As each of x_5, x_6, x_7, y_5, y_6, and y_7 is 2 modulo 4, each is either 2 or 6 modulo 8. If $x_j \equiv y_j \pmod 8$ for some $j \in \{5, 6, 7\}$, then by setting $z = x_j$ in (3), we see that $2^9|C$, and we are done.

Hence, we only need to consider the case that each of x_5, x_6, and x_7 is congruent modulo 8, each of y_5, y_6, and y_7 is congruent modulo 8, and $x_5 \not\equiv y_5 \pmod 8$. As a consequence, one of the sums $x_5 + x_6 + x_7$ or $y_5 + y_6 + y_7$ is equivalent to $2 + 2 + 2 \equiv 6 \pmod 8$ and the other is $6 + 6 + 6 \equiv 2 \pmod 8$. Further, since $(7, 3)$ and $(8, 5)$ are points on the Newton polygon of $f(z)$ and on the Newton polygon of $g(z)$, we obtain from Lemma 2 that

$$x_8 \equiv y_8 \equiv 4 \pmod 8.$$

Further, since the right-most points of the Newton polygons are on or above $(9, 8)$, by Lemma 2 we have

$$x_9 \equiv y_9 \equiv 0 \pmod 8.$$

Since $x_5 + x_6 + x_7 \not\equiv y_5 + y_6 + y_7 \pmod 8$, $x_8 \equiv y_8 \pmod 8$, and $x_9 \equiv y_9 \pmod 8$, we obtain that

$$x_5 + x_6 + x_7 + x_8 + x_9 \not\equiv y_5 + y_6 + y_7 + y_8 + y_9 \pmod 8.$$

This contradicts (4) in Lemma 5 with $t = 4$, $n = 9$ and $k = 1$. Thus, we are done in this case.

4 Lower bound for $\nu_2\left(\overline{C}_8\right)$

In this section, we show that $2^6|C$. Recall, with $n = 8$, we know $2^9 \nmid C$. For possible future analysis, we show in all but one case of conditions on $X = [x_1, \ldots, x_8]$ and $Y = [y_1, \ldots, y_8]$ that we consider, one has $2^8|\overline{C}_8$.

As before, we work with (3), and set $n = 8$ and $C = C_n$. Recall $f(z)$ and $g(z)$ have been translated, if necessary, so that $a_0 \neq 0$, $b_0 \neq 0$ and k_1, k_1', k_2, and k_2' are as before. Thus, $k_1' = k_1 \leq 4$, $k_2' \geq \lceil (8 - k_1')/2 \rceil \geq 2$ and $k_2' \geq k_2$. Since here the lists X and Y have eight elements, $k_1 + k_2 \leq 8$ and $k_1' + k_2' \leq 8$.

Case 1. $k_1' = 4$ and $k_2' = 4$

From Corollary 3, we have $k_2 \equiv k_2' \pmod 4$. Thus, either $k_2 = 0$ or $k_2 = 4$. In the second case, letting $z = 2$ in (3) shows $2^8|C$, as we want. So suppose $k_2 = 0$. In this case, the edges of the Newton polygon of $f(z)$ with positive slope have slope ≥ 2. In particular, this implies

$$\nu_2(a_{8-j}) \geq 2(j - 4) \quad \text{for } 5 \leq j \leq 8.$$

As the points $\big(j, \nu_2(a_{8-j})\big)$ on S_1 and $\big(j, \nu_2(b_{8-j})\big)$ on S_2 agree for $0 \leq j \leq 7$, we deduce

$$\nu_2(b_{8-j}) \geq 2(j - 4) \quad \text{for } 5 \leq j \leq 7. \tag{6}$$

Define $u_j \in \mathbb{Z}$ by the equation

$$(z - y_5)(z - y_6)(z - y_7)(z - y_8) = \sum_{j=0}^{4} u_j z^j.$$

Next, we obtain information on the 2-adic values of the u_j. As $y_j \equiv 2 \pmod{4}$ for $5 \le j \le 8$, we have

$$u_0 = y_5 y_6 y_7 y_8 \implies \nu_2(u_0) = 4.$$

Also, u_1 is the sum of 4 terms that are exactly divisible by 8, so $\nu_2(u_1) \ge 4$. Assume $\nu_2(u_1) = 4$. We make use of the congruence

$$(z - y_1)(z - y_2)(z - y_3)(z - y_4) \equiv (z+1)^4 \equiv z^4 + 1 \pmod{2}. \tag{7}$$

Thus, the product on the left when expanded is a quartic with odd constant term and an odd coefficient for z^4 but otherwise has even coefficients. Thus, there are integers r and s satisfying

$$b_1 = u_1(2r+1) + u_0(2s).$$

Since $\nu_2(u_0) = \nu_2(u_1) = 4$, we deduce $\nu_2(b_1) = 4$. This contradicts (6) with $j = 7$. Thus,

$$\nu_2(u_1) \ge 5.$$

Writing $y_j = 2(2y'_j + 1)$ for $5 \le j \le 8$, we see that

$$u_1 = -2^3(2y'_5+1)(2y'_6+1)(2y'_7+1)(2y'_8+1)\left(\frac{1}{2y'_5+1} + \frac{1}{2y'_6+1} + \frac{1}{2y'_7+1} + \frac{1}{2y'_8+1}\right).$$

We note that every odd square is $1 \pmod{8}$. In particular, $(2y'_k+1)^2 \equiv 1 \pmod{8}$. Therefore, for each $k \in \{5, 6, 7, 8\}$, we have

$$\begin{aligned}
&(2y'_5+1)(2y'_6+1)(2y'_7+1)(2y'_8+1)\frac{1}{2y'_k+1} \\
&\qquad \equiv (2y'_5+1)(2y'_6+1)(2y'_7+1)(2y'_8+1)\frac{1}{2y'_k+1} \cdot (2y'_k+1)^2 \\
&\qquad \equiv (2y'_5+1)(2y'_6+1)(2y'_7+1)(2y'_8+1)(2y'_k+1) \pmod{8}.
\end{aligned}$$

We deduce that

$$-\frac{u_1}{2^3} \equiv (2y'_5+1)(2y'_6+1)(2y'_7+1)(2y'_8+1)\sum_{j=5}^{8}(2y'_j+1) \pmod{8}.$$

Since $\nu_2(u_1) \ge 5$, we deduce that the last sum above must be divisible by 4. Hence,

$$u_3 = -y_5 - y_6 - y_7 - y_8 = -2\sum_{j=5}^{8}(2y'_j+1) \implies \nu_2(u_3) \ge 3.$$

Observe that

$$u_3^2 = 2^2 \sum_{j=5}^{8} (2y_j' + 1)^2 - 2u_2. \tag{8}$$

Since

$$\sum_{j=5}^{8} (2y_j' + 1)^2 \equiv 4 \pmod 8,$$

we see that $2^2 \sum_{j=5}^{8} (2y_j' + 1)^2$ is exactly divisible by 2^4. On the other hand, $\nu_2(u_3) \geq 3$ implies u_3^2 is divisible by 2^6. Hence, (8) implies

$$\nu_2(u_2) = 3.$$

From (7), there exist integers r, s and t such that

$$b_2 = u_2(2r + 1) + u_1(2s) + u_0(2t).$$

The values and estimates obtained above on $\nu_2(u_j)$, with $j \in \{0, 1, 2\}$, imply now that $\nu_2(b_2) = 3$. This contradicts (6) with $j = 6$, completing this case.

Case 2. $k_1' \leq 3$

We can suppose $k_1' \geq 1$ (see Case 1.2.3 of the previous section). Since $k_1' \leq 3$, we obtain $k_2' \geq \lceil (8 - 3)/2 \rceil = 3$. Suppose first that $k_2' < 8 - k_1'$. Since the points $\big(j, \nu_2(a_{8-j})\big)$ on S_1 and $\big(j, \nu_2(b_{8-j})\big)$ on S_2 agree for $0 \leq j \leq 7$, we deduce that $k_2 = k_2'$. In this case, letting $z = 2$ in (3), we see that $2^8 | C$, as we want. Now, suppose $k_2' = 8 - k_1'$. From Corollary 3, we have $k_2 \equiv k_2' \pmod 4$. Hence, $k_2 \geq 1$ and, in particular, $x_{k_1'+1} \equiv 2 \pmod 4$. Let $z = x_{k_1'+1}$ in (3). As $f(z) = 0$ and $g(z)$ is divisible by 2^{10}, we get $2^{10} | C$, contradicting that $2^9 \nmid C$.

Case 3. $k_1' = 4$ and $k_2' < 4$

Given that $k_1' + k_2' < 8$ in addition to knowing $k_1 = k_1'$ and $k_2' \geq k_2$, we deduce $k_1 + k_2 \leq k_1 + k_2' = k_1' + k_2' < 8$. Since the points $\big(j, \nu_2(a_{8-j})\big)$ on S_1 and $\big(j, \nu_2(b_{8-j})\big)$ on S_2 agree for $0 \leq j \leq 7$, we conclude that $k_2 = k_2'$ in this case. Note that $k_2' \geq \lceil (8 - 4)/2 \rceil = 2$. Setting $z = 2$, one checks in this case that $2^{8+k_2'-k_1'}$ divides C. As $8 + k_2' - k_1' \geq 8 + 2 - 4 = 6$, we obtain $2^6 | C$ in this case, giving us the desired conclusion.

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