Newton polygons and
the Prouhet-Tarry-Escott problem

by Michael Filaseta and Maria Markovich
Department of Mathematics
University of South Carolina
Columbia, SC 29063

1 Introduction

For \( n \geq 2 \), we consider two lists of integers

\[ X = [x_1, x_2, \ldots, x_n] \quad \text{and} \quad Y = [y_1, y_2, \ldots, y_n], \]

where, for this section only, we view these as ordered so that \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( y_1 \leq y_2 \leq \cdots \leq y_n \). We also require \( x_j \neq y_j \) for at least one \( j \in \{1, 2, \ldots, n\} \). The Prouhet-Tarry-Escott problem (the PTE problem) asks for such \( X \) and \( Y \) satisfying

\[
\sum_{i=1}^{n} x_i^e = \sum_{i=1}^{n} y_i^e \quad \text{for} \ e = \{1, 2, \ldots, k\}
\]  

(1)

where \( k \) is an integer in the interval \([2, n-1]\). If \( X \) and \( Y \) satisfy (1) then the pair is called a solution of the PTE problem, denoted as \( X =_k Y \). A solution is ideal if \( k = n - 1 \). The significance of the case \( k = n - 1 \) is that with \( X \) and \( Y \) distinct as required above, it is impossible for (1) to hold if \( k \geq n \). Thus, the largest possible value for \( k \) in (1) is \( n - 1 \).

Literature on the PTE problem is extensive. The problem is a focus of an entire chapter (Chapter 24) of L. E. Dickson’s classical volumes “History of the Theory of Numbers” [9] and numerous early references can be found there. The problem is also discussed in G. H. Hardy and E. M. Wright’s well-known “An Introduction to the Theory of Numbers” [12], undoubtedly in part due to Wright’s own interest in the problem (cf. [21, 22, 23]). We note that for the first half of the twentieth century, the problem was referred to as the Tarry-Escott problem, until Wright [22] pointed out that E. Prouhet [17] first discussed the problem in 1851. A few of the more recent investigations on the PTE problem include [4, 5, 8, 14, 18]. Interesting work on generalizations of the PTE problem can be found in [1, 6]. For applications arising from the PTE problem see [2, 11, 13, 16, 19].

\textit{2000 Mathematics Subject Classification:} 11D72, 11B75, 11D41, 11P05.

The first author is grateful to the National Security Agency for funding during research for this paper.
An important open problem in the area is a conjecture of Wright [21] that for every natural number \( n \geq 3 \), an ideal solution exists. Despite its long history, ideal solutions are only known to exist for \( 3 \leq n \leq 10 \) and \( n = 12 \). In particular, no ideal solution is known for \( n = 11 \).

To help formulate further discussion, we note that the following result and its corollary are fairly simple consequences of properties of elementary symmetric functions (see [3, 4]).

**Lemma 1.** Let \( n \) and \( k \) be integers with \( 1 \leq k < n \). Let \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) denote arbitrary integers. The following are equivalent:

\[
\begin{align*}
&\bullet \quad \sum_{i=1}^{n} x_i^e = \sum_{i=1}^{n} y_i^e, \quad \text{for } e \in \{1, 2, \ldots, k\}, \\
&\bullet \quad \deg \left( \prod_{i=1}^{n} (z - x_i) - \prod_{i=1}^{n} (z - y_i) \right) \leq n - k - 1, \\
&\bullet \quad (z - 1)^{k+1} \mid \left( \sum_{i=1}^{n} z^{x_i} - \sum_{i=1}^{n} z^{y_i} \right).
\end{align*}
\]

**Corollary 1.** The lists \( X = [x_1, \ldots, x_n] \) and \( Y = [y_1, \ldots, y_n] \) give an ideal PTE solution if and only if

\[
\prod_{i=1}^{n} (z - x_i) - \prod_{i=1}^{n} (z - y_i) = C
\]

for some real constant \( C \).

In this paper, we will view ideal PTE solutions as being lists \( X \) and \( Y \) satisfying (2). For computational reasons (see [4, 7, 18]), information on possible values of \( C \) and, in particular, on the factorization of \( C \) given (2), has played an important role in arriving at examples of ideal PTE solutions. As \( C \) depends on \( n \), \( X \) and \( Y \), we define, for \( X =_{n-1} Y \), the constant

\[
C_n = C_n(X, Y) = \prod_{i=1}^{n} (z - x_i) - \prod_{i=1}^{n} (z - y_i).
\]

We clarify that what is of interest here then is the value of

\[
\overline{C}_n = \prod_{j=1}^{\infty} p_j^{e_j},
\]

where

\[
e_j = \min \{ e : p_j^e \| C_n(X, Y) \text{ for some } X \text{ and } Y \text{ as above with } X =_{n-1} Y \}.
\]

In other words, \( \overline{C}_n \) can be viewed as the greatest common divisor over all constants \( C_n(X, Y) \) where \( X \) and \( Y \) vary over distinct ordered lists of \( n \) integers satisfying \( X =_{n-1} Y \). So we would like to know, for a given \( n \), how \( \overline{C}_n \) factors.

With the notation above, we state the following result that plays a role throughout the paper; it is an easy consequence of Corollary 1 or Lemma 1.
Corollary 2. Let \( a \in \mathbb{Z} \). The pair of lists \( X = [x_1, \ldots, x_n] \) and \( Y = [y_1, \ldots, y_n] \) form an ideal PTE solution if and only if the pair of lists \( X' = [x_1 + a, \ldots, x_n + a] \) and \( Y' = [y_1 + a, \ldots, y_n + a] \) form an ideal PTE solution. Furthermore, if these are ideal solutions, then \( C_n(X, Y) = C_n(X', Y') \).

The values of \( C_n \) for \( 3 \leq n \leq 7 \) are known (see [7]):

\[
\begin{align*}
C_3 & = 2^2 \\
C_4 & = 2^2 \cdot 3^2 \\
C_5 & = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \\
C_6 & = 2^5 \cdot 3^2 \cdot 5^2 \\
C_7 & = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11.
\end{align*}
\]

In this paper, we pay particular attention to ideal solutions of sizes 8 and 9. For these, according to [7], it is known that

\[
\begin{align*}
C_8 & = 2^{e_1} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13, \quad \text{where} \ 4 \leq e_1 \leq 8 \\
C_9 & = 2^{e_2} \cdot 3^{e_3} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \cdot 66, \quad \text{where} \ 7 \leq e_2 \leq 9, \ 3 \leq e_3 \leq 4 \\
& \quad 0 \leq e_j \leq 1, \text{ for } j \in \{4, 5, 6\}.
\end{align*}
\]

There are two noteworthy examples that pertain to this paper. L. E. Dickson [9] reports that, in 1913, G. Tarry [20] observed that

\[
(x^2 - 5^2)(x^2 - 14^2)(x^2 - 23^2)(x^2 - 42^2) - (x^2 - 2^2)(x^2 - 7^2)(x^2 - 17^2)(x^2 - 25^2) = C
\]

where \( C = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \). According to E. M. Wright [23], in 1942, A. Létac [15] gave the example

\[
\begin{align*}
\end{align*}
\]

\[
= 337742503382400 = 2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 \cdot 41.
\]

These imply

\[
\nu_2(C_8) \leq 8 \quad \text{and} \quad \nu_2(C_9) \leq 9,
\]

where \( \nu_2(m) \) refers to the 2-adic value of \( m \), that is the largest integer \( j \) for which \( 2^j \mid m \). These somewhat old examples then give the upper bounds described above for \( e_1 \) and \( e_2 \).

Our interest in this paper is to explain how the classical theory of Newton polygons can be used to obtain information about the 2-adic values of \( C_n \). In particular, we show that \( \nu_2(C_9) = 9 \). For \( n = 8 \), we only provide the inequality \( \nu_2(C_8) \geq 6 \).

The arguments we give take advantage of working modulo small powers of 2 and substituting values for \( z \) in (2) to show smaller values for \( \nu_2(C_9) \) and \( \nu_2(C_8) \) cannot exist. We have found the example

\[
X = [31914804930538, 392011859134314, 414199788923609],
\]
and

\[ Y = [226375709153429, 382003430459158, 502458387218286, 690280771238587, 750383096702563, 764464731978500, 790357673966989, 870082337037308] \]

which has the property that

\[ \prod_{i=1}^{8} (z - x_i) - \prod_{i=1}^{8} (z - y_i) \equiv 954668492881984 \pmod{2^{50}}. \]

Of interest here is that the number 954668492881984 is exactly divisible by 2^6. Thus, there is no real hope that working modulo small powers of 2 will enable one to show 2^7 must divide \( C \) in (2). Further, substituting any \( z \in \mathbb{Z} \) into the expression on the left above results in an integer exactly divisible by 2^6, so such substitutions will not provide us with a means to show 2^7 must divide \( C \). Perhaps examples like the above exist for the obvious reason that 2^6 \( \parallel \) \( C \), and an appropriate example, different from the one of Tarry’s indicated above, is needed then to show that \( \nu_2(\overline{C}_8) = 6 \).

The example above raises some natural questions. Is it possible to show that a 2-adic ideal solution exists for the PTE problem for every \( n \geq 3 \)? Let \( p \) be a prime. Does a \( p \)-adic ideal solution necessarily exist for \( n = 11 \)? Is it possible to have a \( p \)-adic solution to

\[ \prod_{i=1}^{n} (z - x_i) - \prod_{i=1}^{n} (z - y_i) = C, \]

for which \( \nu_p(C) < \nu_p(\overline{C}_n) \), where \( \nu_p \) is the usual \( p \)-adic valuation and \( n \) is some integer \( \geq 3 \)?

### 2 Further preliminaries

We write

\[ f(z) = \prod_{j=1}^{n} (z - x_j) = \sum_{j=0}^{n} a_j z^j \quad \text{and} \quad g(z) = \prod_{j=1}^{n} (z - y_j) = \sum_{j=0}^{n} b_j z^j \]

where \( x_j, y_j \in \mathbb{Z} \) are chosen so that

\[ f(z) - g(z) = C_n \]

and so that the exact power of 2 dividing \( C_n \) is equal to the exact power of 2 dividing \( \overline{C}_n \). Thus, by Corollary 1, we have that \( X = [x_1, \ldots, x_n] \) and \( Y = [y_1, \ldots, y_n] \) is an ideal solution. We write \( C = C_n \), where \( n \) should be clear from the context.
For fixed \( n \), we consider the two sets of points in the extended plane

\[
S_1 = \{(j, \nu_2(a_{n-j})) : 0 \leq j \leq n\} \quad \text{and} \quad S_2 = \{(j, \nu_2(b_{n-j})) : 0 \leq j \leq n\}.
\]

Since \( f(z) - g(z) = C \), a constant, we see that \( a_{n-j} = b_{n-j} \) for \( 0 \leq j \leq n - 1 \). Thus, \( S_1 \) and \( S_2 \) have at least \( n \) of \( n + 1 \) points in common.

Recalling Corollary 2, we translate \( f(z) \) and \( g(z) \) by the same translation, if necessary, so that \( a_0 \neq 0 \) and \( b_0 \neq 0 \). Thus, \( \nu_2(a_0) \neq +\infty \) and \( \nu_2(b_0) \neq +\infty \). Note that (3) still holds. This ensures that the right-most points \((n, \nu_2(a_0))\) and \((n, \nu_2(b_0))\), which may differ in \( S_1 \) and \( S_2 \), are in the finite plane.

We will be interested in Newton polygons, and in particular to a result that goes back to work of G. Dumas [10].

**Definition 1.** Let \( F(z) = \sum_{j=0}^{n} c_j z^j \in \mathbb{Z}[z] \) with \( c_0 c_n \neq 0 \). Let \( p \) be a prime. For \( j \in \{0, \ldots, n\} \), we define \( x_j = j \) and define \( y_j = \nu_p(c_{n-j}) \). We consider the lower edges along the convex hull of the points in \( S = \{(x_0, y_0), \ldots, (x_n, y_n)\} \). The polygonal path formed by these edges is called the Newton polygon associated with \( F(z) \) with respect to \( p \).

Thus, the Newton polygon of \( f(z) \) with respect to the prime 2 is the lower convex hull of the points in \( S_1 \), and the Newton polygon of \( g(z) \) with respect to 2 is the lower convex hull of the points in \( S_2 \). Note that the slopes of the edges of the Newton polygons increase from left to right. We state next an important property of Newton polygons based on the set-up in this paper.

**Lemma 2.** The Newton polygons of \( f(z) \) and \( g(z) \) will each pass through \( n + 1 \) lattice points (including the endpoints), which we denote respectively as

\[
T_1 = \{(j, t_j) : 0 \leq j \leq n\} \quad \text{and} \quad T_2 = \{(j, t'_j) : 0 \leq j \leq n\}.
\]

After possibly rearranging the \( x_j \) and \( y_j \), we have \( 2^{t_j-t_j-1} \) exactly divides \( x_j \) and \( 2^{t'_j-t'_j-1} \) exactly divides \( y_j \) for each \( j \in \{1, 2, \cdots, n\} \).

This lemma follows directly from a theorem of Dumas [10] which asserts that the Newton polygon of a product of two polynomials with respect to a prime \( p \) can be obtained by translating the edges of the Newton polygons for each polynomial with respect to \( p \). Since \( f(z) \) and \( g(z) \) are a product of \( n \) linear factors, we have that the Newton polygons associated with \( f(z) \) and \( g(z) \) each consists of \( n \) line segments translated so that \( n + 1 \) lattice points (including endpoints) are along its edges. Each translated segment will have the \( x \)-coordinates of its endpoints differing by 1.

As a consequence of Lemma 2, the slope of each edge of the Newton polygon of \( f(z) \) and \( g(z) \) is an integer. In the last statement of Lemma 2, observe that the values \( \nu_2(x_j) \) and \( \nu_2(y_j) \) are increasing as \( j \) ranges from 1 to \( n \). We will want to use such an ordering throughout the remainder of the paper. In particular, the values of the \( x_j \) and the values of the \( y_j \) themselves are not necessarily increasing as in the introduction.

To illustrate, we consider \( n = 9 \) and take the example of A. Létac [15] mentioned in the introduction, so

\[
X = [1, 25, 31, 87, 134, 158, 182, 198, 84] \quad \text{and} \quad Y = [113, 169, 175, 199, 2, 18, 42, 66, 116],
\]
where we have taken an ordering of the $x_j$ and $y_j$ corresponding to the last statement in Lemma 2. In this case,

$$f(z) \equiv g(z) \equiv z^9 + 124z^8 + 70z^7 + 24z^6 + 33z^5 + 12z^4 + 72z^3 + 32z^2 + 80z + 64 \text{ (mod 128)},$$

so that the Newton polygons of $f(z)$ and of $g(z)$ with respect to 2 look the same and are as shown in Figure 1.

![Newton polygon for A. Létac's example](image)

Figure 1: Newton polygon for A. Létac’s example

The solid circles represent the points of $S_1$ and $S_2$ with the bottom left-hand endpoint equal to $(0, 0)$ in each case (since the polynomials are monic). The open circles refer to the lattice points in $T_1$ and $T_2$ as mentioned in Lemma 2. Thus, for this example,

$$T_1 = T_2 = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 1), (6, 2), (7, 3), (8, 4), (9, 6)\}.$$

As implied by Lemma 2, the height differences between two consecutive lattice points in $T_1$ indicates that there are exactly four odd $x_j$’s, four $x_j$’s that are exactly divisible by 2, and one $x_j$ exactly divisible by 4. As $T_1 = T_2$, the $y_j$’s satisfy analogous conditions. We note that despite this example, in general, unlike $S_1$ and $S_2$ which have all but possibly their right-most points in common, the points other than $(0, 0)$ belonging to $T_1$ and $T_2$ can be different.

**Lemma 3.** If the points $(n, \nu_2(a_0))$ in $S_1$ and $(n, \nu_2(b_0))$ in $S_2$ are distinct and

$$k = \min\{ \nu_2(a_0), \nu_2(b_0) \},$$

then $2^k \parallel C$.

**Proof.** Since $C = a_0 - b_0$ and $\nu_2(a_0) \neq \nu_2(b_0)$, we see that

$$\nu_2(C) = \nu_2(a_0 - b_0) = \min\{ \nu_2(a_0), \nu_2(b_0) \} = k.$$

Thus, $2^k \parallel C$. □

We develop some notation that we will be using in the subsequent sections. Let $k_1$ be the number of odd $x_j$ and $k'_1$ be the number of odd $y_j$; thus, the 2-valuation of each of these $x_j$ and $y_j$ is equal to 0. Further, we let $k_2$ be the number of $x_j$ which are congruent to 2 (mod 4) and $k'_2$ be
the number of $y_j$ that are congruent to 2 $(\mod 4)$; thus, the $2$-valuation of each of these $x_j$ and $y_j$ is equal to 1.

By translating $f(z)$ and $g(z)$ by 1 (or some odd number to guarantee that $a_0$ and $b_0$ are not equal to 0), we may suppose $k_1' \leq \lceil n/2 \rceil$. Furthermore, we may now translate by 2 (or some other number that is congruent to 2 $(\mod 4)$) if needed to obtain that $k_2' \geq \lfloor (n - k_1')/2 \rfloor$ of the $y_j$ are congruent to 2 $(\mod 4)$.

Using the following proposition from [4], we deduce that if $C$ is even, then $k_1 = k_1'$.

**Lemma 4.** Let $[x_1, \ldots, x_n] =_{n-1} [y_1, \ldots, y_n]$ be two lists of integers that constitute an ideal PTE solution, and suppose that a prime $p$ divides the constant $C$ associated with this solution. Then we can reorder the integers $y_i$ so that

$$x_j \equiv y_j \pmod{p} \quad \text{for } j \in \{1, \ldots, n\}.$$  

As noted, we can deduce now that the number of odd $x_j$ must equal the number of odd $y_j$, that is, $k_1 = k_1'$. Further, we can interchange the roles of $f(z)$ and $g(z)$, if necessary, so that $k_2' \geq k_2$. Since there are $n$ elements in the lists $X$ and $Y$, it must be the case that $k_1 + k_2 \leq n$ and $k_1' + k_2' \leq n$.

Before ending this section, we establish the following.

**Lemma 5.** Let $n \geq 8$. Suppose $[x_1, \ldots, x_n] =_{n-1} [y_1, \ldots, y_n]$. For $1 \leq j \leq n$, let $x_j$ and $y_j$ be such that $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ are odd and otherwise $x_j$ and $y_j$ are even. Then

$$x_1^k + \cdots + x_t^k \equiv y_1^k + \cdots + y_t^k \pmod{16}, \quad \text{for } k \geq 1.$$  

and

$$x_{t+1}^k + \cdots + x_n^k \equiv y_{t+1}^k + \cdots + y_n^k \pmod{16}, \quad \text{for } k \geq 1. \quad (4)$$

**Proof.** Since $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ are odd, we obtain

$$x_j^4 \equiv y_j^4 \equiv 1 \pmod{16}, \quad \text{for } 1 \leq j \leq t.$$  

Thus,

$$x_1^k + \cdots + x_t^k \equiv x_1^{k+4} + \cdots + x_t^{k+4} \pmod{16}$$  

and

$$y_1^k + \cdots + y_t^k \equiv y_1^{k+4} + \cdots + y_t^{k+4} \pmod{16}.$$  

As $x_j^{k+4} \equiv y_j^{k+4} \equiv 0 \pmod{16}$ for $t + 1 \leq j \leq n$, we deduce that

$$x_1^k + \cdots + x_t^k \equiv x_1^{k+4} + \cdots + x_t^{k+4} \equiv x_1^{k+4} + \cdots + x_n^{k+4}$$  

$$\equiv y_1^{k+4} + \cdots + y_n^{k+4} \equiv y_1^{k+4} + \cdots + y_t^{k+4} \equiv y_1^k + \cdots + y_t^k \pmod{16},$$

provided $1 \leq k + 4 \leq n - 1$. Since $n \geq 8$, the above holds for $1 \leq k \leq 3$. On the other hand,

$$x_1^k + \cdots + x_n^k = y_1^k + \cdots + y_n^k \quad \text{for } 1 \leq k \leq 3.$$  

Hence,

$$x_{t+1}^k + \cdots + x_n^k \equiv y_{t+1}^k + \cdots + y_n^k \pmod{16} \quad \text{for } 1 \leq k \leq 3.$$  

The lemma follows since for $k \geq 4$, both sides of the congruence in (4) are divisible by 16.  

Corollary 3. Let \( n \geq 8 \). Suppose \([x_1, \ldots, x_n] =_{n-1} [y_1, \ldots, y_n]\). Let \( k_1, k_1', k_2 \) and \( k_2' \) be as above. Then \( k_2 \equiv k_2' \pmod{4} \).

Proof. Recall \( k_1 = k_1' \). From Lemma 5, we have
\[
x_{k_1+1}^2 + x_{k_1+1}^2 + \cdots + x_n^2 \equiv x_{k_1+1}^2 + x_{k_1+1}^2 + \cdots + x_n^2 \pmod{16}.
\]
As an even integer \( m \) squared is 4 modulo 16 if \( m \equiv 2 \pmod{4} \) and otherwise is 0 modulo 16, the above congruence can be rewritten as \( 4k_2 \equiv 4k_2' \pmod{16} \). The result follows. \( \square \)

3 The 2-adic value of \( \overline{C}_9 \)

Recall that it is known that \( 2^7 | \overline{C}_9 \) and \( 2^{10} \nmid \overline{C}_9 \). Our goal in this section is to increase the lower bound of the valuation of 2 in \( \overline{C}_9 \). With the aid of Newton polygons, we establish \( 2^9 | \overline{C}_9 \) from which we can deduce that \( 2^9 \| \overline{C}_9 \).

We make use of the notation in the previous section with \( n = 9 \) and deal with two cases, each involving multiple subcases, depending on the values of \( k_1' \) and \( k_2' \).

Case 1. \( k_1' + k_2' = 9 \)

In this case, we are assuming that there are no elements in the list \( Y \) that are congruent to 0 \( \pmod{4} \). We consider possibilities for the Newton polygon of \( f(z) \). From Lemma 4, we know that \( k_1 = k_1' \) odd \( x_j \)'s are in the list \( X \). We recall that \( k_2 \leq k_2' \), which implies that \( X \) contains at most \( k_2' \) elements that are divisible by 2 and not 4. Combining these facts, we have that each point \((j, \nu_2(a_{9-j}))\) in \( S_1 \) is on or above the corresponding point \((j, \nu_2(b_{9-j}))\) in \( S_2 \).

Case 1.1. \( k_2 = k_2' \)

Recall that we have translated \( f(z) \) and \( g(z) \) so that \( k_1' \leq |n/2| = |9/2| = 4 \). Therefore, in this subcase, \( k_2 \) and \( k_2' \) are both greater than or equal to 5. Substituting \( z = 2 \) in (3), we obtain
\[
\prod_{j=1}^{9} (2 - x_j) - \prod_{j=1}^{9} (2 - y_j) = f(2) - g(2) = C,
\]
where at least five of the \( x_j \)'s and at least five of the \( y_j \)'s are 2 modulo 4. Thus, \( 2^{10} \) divides each product, and therefore, their difference. This implies a contradiction, since \( 2^{10} \nmid C \). In other words, it is impossible for \( f(z) - g(z) = C \) with \( \nu_2(C) = \nu_2(\overline{C}_9) \) in this case.

Case 1.2. \( k_2 < k_2' \)

In this subcase, \( X \) must contain some elements that are congruent to 0 \( \pmod{4} \) but \( Y \) cannot. We deduce that the right-most point of the Newton polygon of \( f(z) \) is above the point \((9, \nu_2(b_0))\). Since these endpoints are distinct, by Lemma 3 we have \( 2^{\nu_2(b_0)} || C \). Since all of the even elements in \( Y \) are congruent to 2 \( \pmod{4} \) (thus have valuation equal to 1 with respect to the prime 2), we have that \( \nu_2(b_0) = k_2' \). In the case under consideration, \( \nu_2(b_0) = k_2' = 9 - k_1' \). Since we know that \( 2^7 | C \), we have \( k_2' \geq 7 \) and \( k_1' \leq 2 \).
Case 1.2.1. \( k_1' = 2 \)

In this case \( k_2' = 7 \). By Corollary 3, we deduce \( k_2 \in \{3, 7\} \). Thus, \( 2 - x_j \) and \( 2 - y_j \) are divisible by 4 for \( 2 \leq j \leq 4 \), and \( 2 - x_j \) and \( 2 - y_j \) are divisible by 2 for \( 5 \leq j \leq 8 \). Letting \( z = 2 \) in (3), we see that \( 2^{10} | C' \), giving a contradiction in this case.

Case 1.2.2. \( k_1' = 1 \)

As \( k_2' = 8 \) in this subcase, Corollary 3 implies \( k_2 \in \{0, 4, 8\} \). If \( k_2 \geq 4 \), then setting \( z = 2 \) in (3) leads to \( 2^{12} | C \), giving a contradiction. We are left with considering \( k_2 = 0 \). Thus, for \( j \in \{2, 3, \ldots, 9\} \), we have \( 4 | x_j \).

Observe that setting \( z = 2 \) in (3) implies \( 2^8 || C \). If we now take \( z = x_1 \) in (3), we obtain

\[
C = f(x_1) - g(x_1) = -g(x_1) = -\prod_{j=1}^{9} (x_1 - y_j).
\]

As \( x_1 - y_j \) is odd for \( 2 \leq j \leq 9 \) and \( 2^8 | C \), we deduce that

\[
x_1 \equiv y_1 \pmod{2^8}.
\]

Next, we use that \( X = [x_1, x_2, \ldots, x_9] \) and \( Y = [y_1, y_2, \ldots, y_9] \) being an ideal PTE solution implies

\[
x_1^4 + x_2^4 + \cdots + x_9^4 = y_1^4 + y_2^4 + \cdots + y_9^4.
\]

Since \( x_1 \equiv y_1 \pmod{2^8} \), we easily obtain

\[
x_2^4 + x_3^4 + \cdots + x_9^4 \equiv y_2^4 + y_3^4 + \cdots + y_9^4 \pmod{2^8} \tag{5}
\]

For \( j \in \{2, 3, \ldots, 9\} \), we can write \( y_j = 2(2y_j' + 1) \) for some \( y_j' \in \mathbb{Z} \). As \( (2y_j' + 1)^4 \equiv 1 \pmod{16} \), we obtain

\[
(2y_2' + 1)^4 + (2y_3' + 1)^4 + \cdots + (2y_9' + 1)^4 \equiv 8 \pmod{16},
\]

from which it follows that \( y_2^4 + y_3^4 + \cdots + y_9^4 \) is exactly divisible by \( 2^7 \). On the other hand, \( 4 | x_j \) for \( j \in \{2, 3, \ldots, 9\} \), so \( x_2^4 + x_3^4 + \cdots + x_9^4 \) is divisible by \( 2^8 \). We obtain a contradiction now from (5), so \( f(z) - g(z) = C \) with \( \nu_2(C) = \nu_2(C_9) \) is impossible in this case.

Case 1.2.3. \( k_1' = 0 \)

From (3),

\[
C_9 = f(0) - g(0) = -\prod_{j=1}^{9} x_j + \prod_{j=1}^{9} y_j
\]

is divisible by \( 2^9 \). This is what we set out to show, so we are done in this case. (Alternatively, one can use that the 18 \( x_j \)’s and \( y_j \)’s cannot all have a common prime divisor \( p \) in (3) if \( \nu_p(C_9) \) is minimal. From this point of view, this subcase cannot occur.)
Case 2. $k'_1 + k'_2 < 9$

Since $k'_1 \leq 4$, we have

$$k'_2 \geq \left\lceil \frac{9 - k'_1}{2} \right\rceil \geq 3.$$  

We also have $k'_2 \geq k_2$. We note the importance of the condition $k'_1 + k'_2 < 9$. This implies $k'_2 < 9 - k'_1$. Hence, $(k'_1, 0)$ and $(k'_1 + k'_2, k'_2)$ are points in $S_2$ with $x$-coordinates < 9. Therefore, $(k'_1, 0)$ and $(k'_1 + k'_2, k'_2)$ are points in $S_1$. Since there are exactly $k_1 = k'_1$ odd $x_j$ and the Newton polygon of $f(z)$ has integer slopes, we deduce that the segment joining $(k'_1, 0)$ and $(k'_1 + k'_2, k'_2)$ is part of the Newton polygon of $f(z)$. In particular, $k_2 \geq k'_2 \geq 3$. Since $k'_2 \geq k_2$, we deduce $k_2 = k'_2 \geq 3$.

Case 2.1. $k'_1 \leq 3$

If $k'_1 \leq 3$, then there are at least six even $x_j$ and six even $y_j$. Out of the six even $x_j$’s and the six even $y_j$’s, at least three $x_j$’s and three $y_j$’s are $2 \pmod{4}$. Thus, setting $z = 2$ in (3), we obtain $2^9 | C$, as desired.

Case 2.2. $k'_1 = 4$

We lastly consider $k'_1 = k_1 = 4$ and $k_2 = k'_2 \geq 3$. Since we are in the case where $k'_1 + k'_2 < 9$ and $k'_1 = 4$, we have $k'_2 < 5$. Thus, either $k_2 = k'_2 = 4$ or $k_2 = k'_2 = 3$.

Case 2.2.1. $k_2 = k'_2 = 4$

If $k_2 = k'_2 = 4$, then out of the five even $x_j$’s and the five even $y_j$’s, there are four $x_j$’s and four $y_j$’s that are $2 \pmod{4}$. Setting $z = 2$ in (3), we obtain $2^9 | C$ and are done as before.

Case 2.2.2. $k_2 = k'_2 = 3$

Recall that the slopes of the Newton polygons of $f(z)$ and $g(z)$ are integers and the slopes increase from left to right. For each of these Newton polygons, the edge with slope 1 ends at the point $(k_1 + k_2, k_2) = (7, 3)$. Thus, the remaining edge(s) to the right have slope at least 2, and therefore, the right-most point on each of the Newton polygons must be on or above $(9, 7)$.

If the right-most points on the Newton polygons, $(9, \nu_2(a_0))$ and $(9, \nu_2(b_0))$, are on or above $(9, 9)$, then we take $z = 0$ in (3) to see that $2^9 | C$. This finishes the argument in this case.

If exactly one of the Newton polygons has the right-most point $(9, 7)$, then we set $z = 2$ in (3) to get $2^8 | C$. However, Lemma 3 implies that $2^7 | C$, a contradiction.

If both of the Newton polygons have right-most endpoint $(9, 7)$, then by setting $z = 4$ in (3), we see that $2^9 | C$, giving us the conclusion we want.

We now know that one of the Newton polygons has right-most point $(9, 8)$, and the other has right-most endpoint either $(9, 8)$ or above $(9, 8)$. If the right-most endpoint is $(9, 8)$ for one of these Newton polygons, then its two right-most edges consist of the segment joining $(7, 3)$ to $(8, 5)$ and the segment joining $(8, 5)$ to $(9, 8)$. In particular, if $(9, 8)$ is the right-most endpoint for both of the
Newton polygons, then \( x_8 \equiv y_8 \equiv 8 \pmod{16} \). Setting \( z = 8 \) in (3) for this case, we obtain \( 2^9 | C \), as desired.

Finally, we consider the case that one of the Newton polygons has right-most endpoint \((9,8)\) and the other Newton polygon has right-most endpoint above \((9,8)\). Recall that the two points \((j, \nu_2(a_{9-j}))\) and \((j, \nu_2(b_{9-j}))\) agree for \( j \in \{0, 1, \ldots, 8\} \). We deduce that \((8,5)\) is a point in either \( S_1 \) or \( S_2 \), and thus in both. Hence the edge joining \((7,3)\) and \((8,5)\) is common to both Newton polygons. As each of \( x_5, x_6, x_7, y_5, y_6, \) and \( y_7 \) is \( 2 \mod 4 \), each is either \( 2 \) or \( 6 \) \( \mod 8 \). If \( x_j \equiv y_j \pmod{8} \) for some \( j \in \{5, 6, 7\} \), then by setting \( z = x_j \) in (3), we see that \( 2^9 | C \), and we are done.

Hence, we only need to consider the case that each of \( x_5, x_6, \) and \( x_7 \) is congruent modulo 8, each of \( y_5, y_6, \) and \( y_7 \) is congruent modulo 8, and \( x_5 \neq y_5 \pmod{8} \). As a consequence, one of the sums \( x_5 + x_6 + x_7 \) or \( y_5 + y_6 + y_7 \) is equivalent to \( 2 + 2 + 2 \equiv 6 \pmod{8} \) and the other is \( 6 + 6 + 6 \equiv 2 \pmod{8} \). Further, since \((7,3)\) and \((8,5)\) are points on the Newton polygon of \( f(z) \) and on the Newton polygon of \( g(z) \), we obtain from Lemma 2 that
\[
x_8 \equiv y_8 \equiv 4 \pmod{8}.
\]
Further, since the right-most points of the Newton polygons are on or above \((9,8)\), by Lemma 2 we have
\[
x_9 \equiv y_9 \equiv 0 \pmod{8}.
\]
Since \( x_5 + x_6 + x_7 \neq y_5 + y_6 + y_7 \pmod{8} \), \( x_8 \equiv y_8 \pmod{8} \), and \( x_9 \equiv y_9 \pmod{8} \), we obtain that
\[
x_5 + x_6 + x_7 + x_8 + x_9 \neq y_5 + y_6 + y_7 + y_8 + y_9 \pmod{8}.
\]
This contradicts (4) in Lemma 5 with \( t = 4, n = 9 \) and \( k = 1 \). Thus, we are done in this case.

## 4 Lower bound for \( \nu_2(\overline{C}_8) \)

In this section, we show that \( 2^6 | C \). Recall, with \( n = 8 \), we know \( 2^9 \nmid C \). For possible future analysis, we show in all but one case of conditions on \( X = [x_1, \ldots, x_8] \) and \( Y = [y_1, \ldots, y_8] \) that we consider, one has \( 2^6 | \overline{C}_8 \).

As before, we work with (3), and set \( n = 8 \) and \( C = C_n \). Recall \( f(z) \) and \( g(z) \) have been translated, if necessary, so that \( a_0 \neq 0, b_0 \neq 0 \) and \( k_1, k_1', k_2, \) and \( k_2' \) are as before. Thus, \( k_1' = k_1 \leq 4, k_2' \geq \lceil (8 - k_1')/2 \rceil \geq 2 \) and \( k_2' \geq k_2 \). Since here the lists \( X \) and \( Y \) have eight elements, \( k_1 + k_2 \leq 8 \) and \( k_1' + k_2' \leq 8 \).

### Case 1. \( k_1' = 4 \) and \( k_2' = 4 \)

From Corollary 3, we have \( k_2 \equiv k_2' \pmod{4} \). Thus, either \( k_2 = 0 \) or \( k_2 = 4 \). In the second case, letting \( z = 2 \) in (3) shows \( 2^8 | C \), as we want. So suppose \( k_2 = 0 \). In this case, the edges of the Newton polygon of \( f(z) \) with positive slope have slope \( \geq 2 \). In particular, this implies
\[
\nu_2(a_{8-j}) \geq 2(j - 4) \quad \text{for } 5 \leq j \leq 8.
\]
As the points \((j, \nu_2(a_{8-j}))\) on \( S_1 \) and \((j, \nu_2(b_{8-j}))\) on \( S_2 \) agree for \( 0 \leq j \leq 7 \), we deduce
\[
\nu_2(b_{8-j}) \geq 2(j - 4) \quad \text{for } 5 \leq j \leq 7.
\]
Define \( u_j \in \mathbb{Z} \) by the equation

\[
(z - y_5)(z - y_6)(z - y_7)(z - y_8) = \sum_{j=0}^{4} u_j z^j.
\]

Next, we obtain information on the 2-adic values of the \( u_j \). As \( y_j \equiv 2 \pmod{4} \) for \( 5 \leq j \leq 8 \), we have

\[
u_2(u_0) = 4.
\]

Also, \( u_1 \) is the sum of 4 terms that are exactly divisible by 8, so \( \nu_2(u_1) \geq 4 \). Assume \( \nu_2(u_1) = 4 \). We make use of the congruence

\[
(z - y_1)(z - y_2)(z - y_3)(z - y_4) \equiv (z + 1)^4 \equiv z^4 + 1 \pmod{2}.
\]

Thus, the product on the left when expanded is a quartic with odd constant term and an odd coefficient for \( z^4 \) but otherwise has even coefficients. Thus, there are integers \( r \) and \( s \) satisfying

\[
b_1 = u_1(2r + 1) + u_0(2s).
\]

Since \( \nu_2(u_0) = \nu_2(u_1) = 4 \), we deduce \( \nu_2(b_1) = 4 \). This contradicts (6) with \( j = 7 \). Thus,

\[
\nu_2(u_1) \geq 5.
\]

Writing \( y_j = 2(2y'_j + 1) \) for \( 5 \leq j \leq 8 \), we see that

\[
u_2(u_1) = 4.
\]

We note that every odd square is 1 \( \pmod{8} \). In particular, \( (2y'_k + 1)^2 \equiv 1 \pmod{8} \). Therefore, for each \( k \in \{5, 6, 7, 8\} \), we have

\[
(2y'_5 + 1)(2y'_6 + 1)(2y'_7 + 1)(2y'_8 + 1) \frac{1}{2y'_k + 1}
\]

\[
\equiv (2y'_5 + 1)(2y'_6 + 1)(2y'_7 + 1)(2y'_8 + 1) \frac{1}{2y'_k + 1} \cdot (2y'_k + 1)^2
\]

\[
\equiv (2y'_5 + 1)(2y'_6 + 1)(2y'_7 + 1)(2y'_8 + 1)(2y'_k + 1) \pmod{8}.
\]

We deduce that

\[
-\frac{u_1}{2^3} \equiv (2y'_5 + 1)(2y'_6 + 1)(2y'_7 + 1)(2y'_8 + 1) \sum_{j=5}^{8} (2y'_j + 1) \pmod{8}.
\]

Since \( \nu_2(u_1) \geq 5 \), we deduce that the last sum above must be divisible by 4. Hence,

\[
u_2(u_3) = -y_5 - y_6 - y_7 - y_8 = -2 \sum_{j=5}^{8} (2y'_j + 1) \implies \nu_2(u_3) \geq 3.
\]
Observe that

$$u_3^2 = 2^2 \sum_{j=5}^{8} (2y'_j + 1)^2 - 2u_2. \tag{8}$$

Since

$$\sum_{j=5}^{8} (2y'_j + 1)^2 \equiv 4 \pmod{8},$$

we see that $2^2 \sum_{j=5}^{8} (2y'_j + 1)^2$ is exactly divisible by $2^4$. On the other hand, $\nu_2(u_3) \geq 3$ implies $u_3^2$ is divisible by $2^6$. Hence, (8) implies

$$\nu_2(u_2) = 3.
$$

From (7), there exist integers $r$, $s$ and $t$ such that

$$b_2 = u_2(2r + 1) + u_1(2s) + u_0(2t).$$

The values and estimates obtained above on $\nu_2(u_j)$, with $j \in \{0, 1, 2\}$, imply now that $\nu_2(b_2) = 3$. This contradicts (6) with $j = 6$, completing this case.

**Case 2. $k'_1 \leq 3$**

We can suppose $k'_1 \geq 1$ (see Case 1.2.3 of the previous section). Since $k'_1 \leq 3$, we obtain $k'_2 \geq \lceil (8 - 3)/2 \rceil = 3$. Suppose first that $k'_2 < 8 - k'_1$. Since the points $(j, \nu_2(a_{8-j}))$ on $S_1$ and $(j, \nu_2(b_{8-j}))$ on $S_2$ agree for $0 \leq j \leq 7$, we deduce that $k_2 = k'_2$. In this case, letting $z = 2$ in (3), we see that $2^8 \mid C$, as we want. Now, suppose $k'_2 = 8 - k'_1$. From Corollary 3, we have $k_2 \equiv k'_2 \pmod{4}$. Hence, $k_2 \geq 1$ and, in particular, $x_{k'_1+1} \equiv 2 \pmod{4}$. Let $z = x_{k'_1+1}$ in (3). As $f(z) = 0$ and $g(z)$ is divisible by $2^{10}$, we get $2^{10} \mid C$, contradicting that $2^9 \nmid C$.

**Case 3. $k'_1 = 4$ and $k'_2 < 4$**

Given that $k'_1 + k'_2 < 8$ in addition to knowing $k_1 = k'_1$ and $k'_2 \geq k_2$, we deduce $k_1 + k_2 \leq k_1 + k'_2 = k'_1 + k'_2 < 8$. Since the points $(j, \nu_2(a_{8-j}))$ on $S_1$ and $(j, \nu_2(b_{8-j}))$ on $S_2$ agree for $0 \leq j \leq 7$, we conclude that $k_2 = k'_2$ in this case. Note that $k'_2 \geq \lceil (8 - 4)/2 \rceil = 2$. Setting $z = 2$, one checks in this case that $2^8 + k'_2 - k'_1$ divides $C$. As $8 + k'_2 - k'_1 \geq 8 + 2 - 4 = 6$, we obtain $2^6 \mid C$ in this case, giving us the desired conclusion.

**References**


