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Starting with gaps between k-free numbers

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We survey various developments in Number Theory that were inspired by classical papers by K. F. Roth [74] and by H. Halberstam and K. F. Roth [38].

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Dedicated to the memories of Felice and Paul Bateman and of Heini Halberstam

1. Introduction

In 1950 and 1952, K. F. Roth and H. Halberstam, respectively, received their doctoral degrees under the direction of T. Estermann, who was to only supervise, some years later, one other doctoral student, R. Vaughan. In 1951, in two consecutive articles in the Journal of the London Mathematical Society, early papers by K. F. Roth [74] and by H. Halberstam and K. F. Roth [38], described an elementary approach associated with gaps between k-free numbers, that is positive integers not divisible by the kth power of a prime where k is a fixed integer ≥ 2. Specifically, in [38], we find the following result, generalizing a result in [74] where the case $k = 2$ is considered.

Theorem 1.1 (H. Halberstam and K. F. Roth, 1951). *Let k be an integer ≥ 2, and fix $\varepsilon > 0$. For $x \geq x_0(k, \varepsilon)$ sufficiently large, there is a k-free number in*

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the interval $(x, x + x^{(1/(2k))+\varepsilon}]$.

In this paper, we begin by describing some elementary approaches to the gap problem for k-free numbers and, in particular, the approach of H. Halberstam and K. F. Roth in [38]. Then we discuss various developments and applications that have come about due in large part to their initial insights. We demonstrate, therefore, the tremendous impact of [74] and [38].

We turn here then to describing an approach for tackling the problem of finding $h = h(x)$ tending to infinity but as small as possible such that, for all x sufficiently large, the interval $(x, x + h]$ contains a k-free number, where k is fixed as in the theorem. The very basic ideas can be described as follows.

- Find upper bounds for the number of integers $m \in (x, x + h]$ for which there exists a prime p such that $p^k | m$.
- Base such bounds on the size of the prime p.
- Using the bounds, show that there are $\leq h - 1$ such $m \in (x, x+h]$ divisible by the kth power of a prime.

Then we can conclude that $(x, x + h]$ contains a k-free number. The estimates we obtain will govern how small we can take h. With some realistic expectations of what we can prove, we will assume $h = h_k(x)$ grows as fast as some fixed power of x that possibly depends on k (as in Theorem 1.1).

With the above in mind, we describe first how to handle small primes, say $p \leq h\sqrt{\log x}$. We want then an upper bound for the number of $m \in (x, x + h]$ that are divisible by p^k for some $p \leq h\sqrt{\log x}$. Let $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and let $\pi(t)$ denote the number of primes $\leq t$. Since $k \geq 2$, we have $\zeta(k) \leq \zeta(2) = \pi^2/6 < 5/3$. A simple application of the Prime Number Theorem now gives that, for x sufficiently large and h tending to infinity faster than an arbitrarily fixed small power of x (possibly depending on k), the number of $m \in (x, x + h]$ that are divisible by p^k for some $p \leq h\sqrt{\log x}$ is

$$\leq \sum_{p \leq h\sqrt{\log x}} \left(\left\lfloor \frac{x+h}{p^k} \right\rfloor - \left\lfloor \frac{x}{p^k} \right\rfloor \right) \leq \sum_{p \leq h\sqrt{\log x}} \left(\frac{h}{p^k} + 1 \right)$$
$$\leq h\big(\zeta(k) - 1\big) + \pi\big(h\sqrt{\log x}\,\big) < \frac{2h}{3}.$$

Recalling that we want to show that there are $\leq h - 1$ such $m \in (x, x + h]$ divisible by the kth power of a prime, it suffices now to show that, as $x \to \infty$, there exist $< h/4$ values of $m \in (x, x + h] \cap \mathbb{Z}$ divisible by p^k for some $p > h\sqrt{\log x}$.

Before continuing, we note that we could have used the above argument to show that there are $o(h)$ integers $m \in (x, x + h]$ divisible by p^k for some p satisfying $\log x < p \leq h\sqrt{\log x}$. Then a sieve argument could have been used to show that the number of $m \in (x, x+h]$ divisible by p^k for some $p \leq \log x$ is asymptotic to $h/\zeta(k)$. This would end up giving an *asymptotic estimate* for the number of k-free integers in an interval $(x, x + h]$.

We now consider primes $p > h\sqrt{\log x}$. For each such p, there is at most one $m \in (x, x+h]$ such that $p^k | m$. Therefore, the number of $m \in (x, x+h]$ divisible by the kth power of a prime $> h\sqrt{\log x}$ is bounded above by the number of primes $p > h\sqrt{\log x}$ for which p^k divides some integer $m \in (x, x+h]$. The idea now is to show that the number of these large primes p with p^k dividing some $m \in (x, x+h]$ is $< h/4$ or perhaps better $o(h)$. So the focus now is on bounding the number of such primes.

Fix a prime $p > h\sqrt{\log x}$. Suppose $m \in (x, x+h]$ is divisible by p^k. Then there is an integer m' such that $m = p^k m'$ implying that $m/p^k \in \mathbb{Z}$. Also,

$$x < m \le x + h \implies \frac{x}{p^k} < m' \le \frac{x}{p^k} + \frac{h}{p^k}.$$

Using the notation

$$\|t\| = \min\{|t-k| : k \in \mathbb{Z}\},$$

we deduce that

$$\left\| \frac{x}{p^k} \right\| \le \frac{h}{p^k}.$$

We clarify for future reference that for $p > h\sqrt{\log x}$ and x large, we have $h/p^k < 1/2$; hence, the nearest integer to x/p^k is m'.

Our arguments from here on out will not use the primality of p. In other words, we can now replace the prime p above with an integer u and estimate the number of $u > h\sqrt{\log x}$ for which

$$\left\| \frac{x}{u^k} \right\| \le \frac{h}{u^k}. \tag{1.1}$$

This then will be an estimate for the number of primes p as above as well, and we will have achieved our goal if we can show the number of such u is $o(h)$. We pause for a moment to examine what one might expect here so as to see that this is a reasonable tactic. Suppose we fix $N \ge h\sqrt{\log x}$, and take an integer $u \in (N, 2N]$ as in (1.1). Treating $\|x/u^k\|$ as if it were a random value in $[0, 1/2)$, one can expect that (1.1) holds with probability $O(h/N^k)$. Then the expected number of integers $u \in (N, 2N]$ for which (1.1) holds is $O(h/N^{k-1})$. Now using the dyadic approach of taking $N = N_j = 2^j h\sqrt{\log x}$ above, we see that we can expect the number of $u > h\sqrt{\log x}$ for which (1.1) holds to be

$$\ll \sum_{j=0}^{\infty} \frac{h}{N_j^{k-1}} \ll \sum_{j=0}^{\infty} \frac{h}{2^{(k-1)j}(h\sqrt{\log x})^{k-1}} \ll \frac{h}{(h\sqrt{\log x})^{k-1}}.$$

Given $k \ge 2$, we see that one can expect the number of such u to be $o(1)$, even smaller than $o(h)$. Setting as our goal then to show the number of such u is $o(h)$ is quite reasonable.

The emphasis in this paper is on the elementary approach given in [74] and [38]. At this point in the argument, however, it is very reasonable to use exponential sum techniques, used also in [74] and mentioned in [38]. For this general approach, one can consult the book [35]. To clarify, the connection with (1.1), we note the following type result is classical and a simple argument can be found in [28].

Theorem 1.2. *Let $\delta \in (0, 1/2)$, and let $f : \mathbb{R} \mapsto \mathbb{R}$ be any function. Let $S \subseteq \mathbb{Z}^+$ with $|S|$ finite. Then for any positive integer $J \le 1/(4\delta)$, we get that*

$$|\{u \in S : ||f(u)|| < \delta\}| \le \frac{\pi^2}{2(J+1)} \sum_{1 \le j \le J} \left| \sum_{u \in S} e\,(jf(u)) \right| + \frac{\pi^2}{4(J+1)} |S|.$$

Taking $S = (N, 2N]$ with N as in the previous paragraph, $f(u) = x/u^k$ and $\delta = h/N^k$, one is then left with choosing J, estimating an exponential sum to bound the number of $u \in (N, 2N]$ satisfying (1.1) and summing over the appropriate N.

We turn to elementary approaches for estimating the number of primes $p \ge h\sqrt{\log x}$ satisfying (1.1) with $u = p$. Simply showing that there cannot be many such p can be accomplished without a lot of effort or ingenuity, at least for h small but relatively large compared to the result in Theorem 1.1. For example, knowing that we are aiming for $h < x$, we have

$$p^k | m \text{ and } m \in (x, x+h] \implies p < \sqrt[k]{2x}.$$

Then $p > h\sqrt{\log x}$ implies we can take $h = \sqrt[k]{2x}/\sqrt{\log x}$ since then each prime $p > h\sqrt{\log x}$ satisfying (1.1) with $u = p$ also satisfies

$$p > h\sqrt{\log x} \ge \sqrt[k]{2x} > p.$$

In other words, with $h = \sqrt[k]{2x}/\sqrt{\log x}$, the number of primes $p > h\sqrt{\log x}$ satisfying (1.1) with $u = p$ is 0.

Simple ideas can be used to further improve on this choice of h. To clarify, here is an easy approach for establishing that one can take $h = x^{2/(2k+1)}$ for x sufficiently large. Set $T = x^{1/(2k+1)}/(2k)$. Assume that there exist primes p_1 and p_2 satisfying

$$h\sqrt{\log x} < p_2 < p_1 < \sqrt[k]{2x}, \qquad p_1 - p_2 < T,$$

and

$$\left\| \frac{x}{p_j^k} \right\| \le \frac{h}{p_j^k} \quad \text{for } j \in \{1, 2\}. \tag{1.2}$$

Since

$$p_1^k - p_2^k = (p_1 - p_2)(p_1^{k-1} + p_1^{k-2}p_2 + \cdots + p_2^{k-1}) < (p_1 - p_2)kp_1^{k-1}$$

and each $p_j > h = x^{2/(2k+1)}$, we deduce that

$$\frac{x}{p_2^k} - \frac{x}{p_1^k} < \frac{(p_1 - p_2)kp_1^{k-1}x}{p_1^k p_2^k} = \frac{(p_1 - p_2)kx}{p_1 p_2^k} < \frac{(p_1 - p_2)kx}{x^{2(k+1)/(2k+1)}} = \frac{(p_1 - p_2)k}{x^{1/(2k+1)}}.$$

Since $p_1 - p_2 < T = x^{1/(2k+1)}/(2k)$, we obtain now that

$$\frac{x}{p_2^k} - \frac{x}{p_1^k} < \frac{1}{2}.$$

On the other hand, with x large, (1.2) implies that each x/p_j^k is at most a distance of $1/4$ from an integer. Since the difference between these numbers is $< 1/2$, we deduce that they are a distance of $1/4$ from the same integer. Recalling the comments before (1.1), if $m_j \in \mathbb{Z}$ is such that $x < p_j^k m_j \le x+h$, then the nearest integer to x/p_j^k is m_j. Thus, we have $m_1 = m_2$. However, since $p_1 > p_2 > h\sqrt{\log x}$, we deduce

$$\begin{aligned} m_1 = m_2 \quad \Longrightarrow \quad h &> |p_1^k m_1 - p_2^k m_2| = (p_1^k - p_2^k) m_1 \\ &\ge \big((p_2+1)^k - p_2^k\big) > k p_2^{k-1} > h, \end{aligned} \tag{1.3}$$

a contradiction. Hence, the primes $p > h\sqrt{\log x}$, satisfying (1.1) with $u = p$ and satisfying that p^k divides some $m \in (x, x+h]$, are separated by a distance of at least $T = x^{1/(2k+1)}/(2k)$. Therefore, the number of primes $p \in \big(h\sqrt{\log x},\, \sqrt[k]{2x}\,\big)$ for which a multiple of p^k lies in $(x, x+h]$ is

$$\ll_k \frac{\sqrt[k]{2x}}{x^{1/(2k+1)}} \ll_k x^{(k+1)/(k(2k+1))}.$$

Observe now that, with our choice of $h = x^{2/(2k+1)}$, this last estimate is $o(h)$, implying that we can obtain a gap result with this value of h as claimed.

The above idea was based on using a difference of two values of the function $f(u) = x/u^k$. A single difference at values of u that are near one another approximates $f'(u)$. Further differencing, approximating higher derivatives, can be used to expand on this simple idea, improving the exponent on x appearing in the value of h further.

The idea of K. F. Roth [74] and H. Halberstam and K. F. Roth [38] is to bypass the iterative procedure of applying more and more differences to approximate higher and higher derivatives by using a Padé approximate for a high order derivative. Taking

$$P(z) = 1 - \binom{2k-1}{1} z + \cdots + (-1)^{k-1} \binom{2k-1}{k-1} z^{k-1} \in \mathbb{Z}[z],$$

one sees that there is a $Q(z)$ in $\mathbb{Z}[z]$ satisfying

$$(1-z)^{2k-1} = P(z) - z^k Q(z). \tag{1.4}$$

With p_1 and p_2 primes $> h\sqrt{\log x}$, we set $z = p_2/p_1$. The polynomials $P(z)$ and $Q(z)$ are of degree $k-1$ so that

$$P_0 = P_0(p_1,p_2) = p_1^{k-1} P(p_2/p_1) \quad \text{and} \quad Q_0 = Q_0(p_1,p_2) = p_1^{k-1} Q(p_2/p_1)$$

are in $\mathbb{Z}$. The substitution $z = p_2/p_1$ in (1.4) gives

$$\begin{aligned} (p_1 - p_2)^{2k-1} &= p_1^k \big(p_1^{k-1} P(p_2/p_1)\big) - p_2^k \big(p_1^{k-1} Q(p_2/p_1)\big) \\ &= p_1^k\, P_0(p_1,p_2) - p_2^k\, Q_0(p_1,p_2). \end{aligned}$$

Thus,

$$\frac{x}{p_2^k}\, P_0(p_1,p_2) - \frac{x}{p_1^k}\, Q_0(p_1,p_2) = \frac{(p_1-p_2)^{2k-1} x}{p_1^k p_2^k}. \tag{1.5}$$

This replaces the direct difference $(x/p_2^k) - (x/p_1^k)$ with a modified difference that involves the integer multipliers P_0 and Q_0.

Considering p_1 and p_2 in an interval $(N, 2N]$ with $p_1 > p_2$ and $N \geq h\sqrt{\log x}$, we suppose now that

$$p_1 - p_2 \leq T = \left(\frac{N^{2k}}{2x}\right)^{1/(2k-1)}.$$

Since P_0 and Q_0 are in $\mathbb{Z}$, (1.2) implies

$$\left\| \frac{x}{p_2^k} P_0 \right\| \leq \frac{hP_0}{p_2^k} \quad \text{and} \quad \left\| \frac{x}{p_1^k} Q_0 \right\| \leq \frac{hQ_0}{p_1^k}. \tag{1.6}$$

Recalling the polynomials $P(z)$ and $Q(z)$ have degree $k-1$, we see that the right-hand sides of the inequalities in (1.6) are each

$$\ll_k \frac{hN^{k-1}}{N^k} \ll_k \frac{h}{N} \ll_k \frac{1}{\sqrt{\log x}}.$$

Thus, with x sufficiently large, the left-hand sides of (1.6) are each $< 1/4$. On the other hand, $p_1 - p_2 \leq T$ implies from (1.5) that the expressions $(x/p_2^k)P_0$ and $(x/p_1^k)Q_0$ differ by $\leq 1/2$. Hence, these expressions are within $1/4$ of the same integer.

We could proceed to try to use (1.5) and (1.6) in a manner similar to our previous argument, but the approach does not give us what we want. Here, another clever idea in [74] and [38] is used. With p_1 and p_2 as above, (1.5) led to us showing that the closest integer to $(x/p_2^k)P_0$ and $(x/p_1^k)Q_0$ is the same. Let m_j denote the integer nearest to x/p_j^k for $j \in \{1, 2\}$. Note that, for each $j \in \{1, 2\}$, we have

$$\frac{x}{p_j^k} > \frac{x}{(\sqrt[k]{2x}\,)^k} = \frac{1}{2} \implies m_j \geq 1. \tag{1.7}$$

Given $\|x/p_2^k\| \leq h/p_2^k$, we deduce that

$$\frac{x}{p_2^k} P_0 = m_2 P_0 + O\left(\frac{hP_0}{p_2^k}\right) = m_2 P_0 + O\left(\frac{1}{\sqrt{\log x}}\right).$$

Thus, the nearest integer to $(x/p_2^k)P_0$ is $m_2 P_0$. Similarly, the nearest integer to $(x/p_1^k)Q_0$ is $m_1 Q_0$, and we deduce

$$m_2 P_0(p_1, p_2) - m_1 Q_0(p_1, p_2) = 0,$$

where we have emphasized here that P_0 and Q_0 depend on p_1 and p_2. Note, however, that each m_j only depends on the corresponding p_j. Now, we suppose that there is a third prime $p_3 \in (N, 2N]$ with $p_1 > p_2 > p_3$ and with all 3 primes within T of each other. Here, we also suppose (1.1) holds with $u = p_3$. The above analysis then leads to the additional equations

$$m_3 P_0(p_1, p_3) - m_1 Q_0(p_1, p_3) = 0$$

and

$$m_3 P_0(p_2, p_3) - m_2 Q_0(p_2, p_3) = 0.$$

Multiplying the first of these last two equations by $P_0(p_2, p_3)$, the second by $P_0(p_1, p_3)$, and subtracting gives

$$m_2 P_0(p_1, p_3) Q_0(p_2, p_3) - m_1 P_0(p_2, p_3) Q_0(p_1, p_3) = 0. \tag{1.8}$$

We view p_1 and p_2 as fixed, and hence m_1 and m_2 as fixed, and p_3 as a variable. The polynomials $P_0(p_1, p_3)$, $Q_0(p_2, p_3)$, $P_0(p_2, p_3)$ and $Q_0(p_1, p_3)$ all have degree $k-1$ in p_3 and their leading coefficients are respectively

$$(-1)^{k-1}\binom{2k-1}{k-1}, \quad 1, \quad (-1)^{k-1}\binom{2k-1}{k-1} \quad \text{and} \quad 1.$$

Recalling (1.3), we see that $m_2 \neq m_1$, so the left-hand side of (1.8) has degree $2k-2$ and leading non-zero coefficient

$$(-1)^{k-1}\binom{2k-1}{k-1}(m_2 - m_1).$$

We deduce now that (1.8) has at most $2k-2$ solutions in p_3. Therefore, there are at most $2k$ primes $p \in (N, 2N]$ all within a distance T from one another.

We deduce then that the number of primes $p \in (N, 2N]$ for which some multiple of p^k is in the interval $(x, x+h]$ is

$$\ll_k \frac{N}{T} \ll_k \frac{N\,(2x)^{1/(2k-1)}}{N^{2k/(2k-1)}} \ll_k x^{1/(2k-1)} N^{-1/(2k-1)}.$$

We use the dyadic approach mentioned earlier considering $N = 2^j h\sqrt{\log x}$ where $j \in \{0, 1, \dots, r\}$ with

$$2^r h < (2x)^{1/k} \le 2^{r+1} h.$$

Then the number of primes $p > h\sqrt{\log x}$ for which there is a multiple of p^k in $(x, x+h]$ is then

$$\ll_k \sum_{j=0}^{r} x^{1/(2k-1)} (2^j h\sqrt{\log x}\,)^{-1/(2k-1)} \ll_k x^{1/(2k-1)} (h\sqrt{\log x}\,)^{-1/(2k-1)}.$$

Observe that with $h = x^{1/(2k)}$, this last expression is $o(h)$ as $x \to \infty$. Hence, the above shows that we can find k-free numbers in the interval $(x, x + x^{1/(2k)}]$ for x sufficiently large, a result slightly stronger than Theorem 1.1. The improvement over [74] and [38] is in our use, taken from [68], of (1.8) to bound the number of possible values of p_3.

Of some significance here is that if we replace z with $1-z$ in (1.4), then we are led to

$$\begin{aligned} z^{2k-1} &= P(1-z) - (1-z)^k Q(1-z) \\ &\implies (1-z)^k = \frac{P(1-z)}{Q(1-z)} - \frac{z^{2k-1}}{Q(1-z)} = \frac{P(1-z)}{Q(1-z)} + O(z^{2k-1}), \end{aligned}$$

as $z \to 0$. Thus, $P(1-z)/Q(1-z)$ is a Padé approximate for $(1-z)^k$, so that the use of (1.4) to obtain Theorem 1.1 is one of many examples of the tremendous impact of the hypergeometric method in analytic and transcendental number theory, a method having origins in the earlier work of A. Thue [85] and C. L. Siegel [82] (cf. [3], [4], [5], [6]-[11], [12]-[13], [14]-[16], [20], [65], [86], [89]-[90]).

2. Powerfree values of polynomials

Having introduced the motivating ideas behind the work of K. F. Roth [74] and H. Halberstam and K. F. Roth [38], we devote most of the remainder of this paper to known applications without proofs. The earliest of these applications that we discuss is due to M. Nair [68,69] (see also [50]). The work of M. Nair [68], already addressed above, allowed a simple alteration in the arguments in [74] and [38] that enables one to remove the ε appearing in the exponent in Theorem 1.1. But much more importantly, M. Nair was able to extend the ideas in [74] and [38] to algebraic number fields and produce some very nice results on powerfree values of polynomials that we discuss next.

Let k be an integer ≥ 2. Let $f(x)$ be an irreducible polynomial in $\mathbb{Z}[x]$ of degree n. Necessarily, we require that $\gcd(f(m) : m \in \mathbb{Z})$ is k-free. For computational purposes, we note that as a consequence of Lagrange's interpolation formula we have

$$\gcd(f(m) : m \in \mathbb{Z}) = \gcd\big(f(m) : m \in \{0, 1, \ldots, n\}\big).$$

Extending the ideas of K. F. Roth [74] H. Halberstam and K. F. Roth [38] to algebraic number fields and making a slight improvement on the approach as noted with (1.8) above, M. Nair [69,68] established the following

Theorem 2.1 (M. Nair, 1976/79). *Let k, $f(x)$ and n be as above. If $k \geq n+1$, then there is a constant $c = c(k, f)$ such that the interval*

$$(x, x+h], \quad \text{where } h = c\, x^{n/(2k-n+1)},$$

contains an integer m for which $f(m)$ is k-free.

Observe that Theorem 2.1 implies Theorem 1.1 by taking $f(x) = x$. A number of related results and improvements ensued. With k, $f(x)$ and n still as above, M. N. Huxley and M. Nair [50] showed that if $k \geq n+1 \geq 3$, one can take $h = c\, x^{n/(2k-n+2)}$. Some time later, the first author [26] showed that if $k \geq n+1 \geq 2$, one can take $h = c\, x^{n/(2k-n+\sqrt{2n}-(1/2))}$. As before, $c = c(k, f)$ is a constant in these results.

Theorem 2.1 gives a natural generalization of Theorem 1.1. However, M. Nair presented another twist in his papers. What if instead of looking for short intervals $(x, x+h]$ for the m satisfying $f(m)$ is k-free, we consider large intervals of this form with h around the size of x? For short intervals, the above gives a result for k-free values of polynomials provided $k \geq n+1$; but M. Nair [69] showed that by considering larger intervals, one can obtain results for k-free values of polynomials for smaller k. The larger h is with $h \leq x$, the smaller his method allowed one to take k. To understand the impact of this approach, we mention briefly the history here.

Improving on work of T. Nagel [67] and G. Ricci [72], P. Erdős [22] established the following.

Theorem 2.2 (P. Erdős, 1953). *Let k, $f(x)$ and n be as above. If $k \geq n-1$, then there exist infinitely many integers m for which $f(m)$ is k-free.*

An important achievement of C. Hooley [45] in 1967 was to improve on Theorem 2.2 by showing that the asymptotic density of integers m for which $f(m)$ is k-free in Theorem 2.2 is positive and can be given explicitly. Using the methods of K. F. Roth [74] H. Halberstam and K. F. Roth [38] in algebraic number fields, M. Nair [69] established

Theorem 2.3 (M. Nair, 1976). *Under the conditions above but with instead $k \geq (\sqrt{2} - (1/2))\, n$, the asymptotic density of integers m for which $f(m)$ is k-free is positive and can be given explicitly.*

C. Hooley's work [45] gives the best known estimates for n small, and for 30 years M. Nair's Theorem 2.3 remained the best known result for large n.

Using a different approach, some recent nice improvements have been made by D. R. Heath-Brown [39,40], who obtained an analogous result for $k \geq (3n+2)/4$, and then slightly further by T. D. Browning [18] (both arguments based on counting integer points on the affine surface $f(x) = y^k z$ by showing that they lie on a small number of curves). In particular, we have the following.

Theorem 2.4 (T. D. Browning, 2011). *Under the conditions above but with $k \geq (3n+1)/4$, there are infinitely many integers m for which $f(m)$ is k-free.*

All the methods in these results extend to obtain information about k-free values of binary forms (cf. [25], [37], [18]). In particular, T. D. Browning has obtained the analogous result for irreducible binary forms of degree n and $k > 7n/16$. We also note that C. Hooley [42,43] has obtained, with yet another approach, similar results for polynomials $f(x,y) \in \mathbb{Z}[x,y]$ that are not necessarily forms.

3. Further progress on gaps between k-free numbers

The first author, as a graduate student, working under the direction of H. Halberstam, studied [74] and [38] and began working on further elementary approaches to gap results for k-free numbers. As part of his dissertation [30], he gave an elementary argument giving an exponentially small improvement on Theorem 1.1. Specifically, he showed that for every $\varepsilon > 0$ and for x sufficiently large, there is a k-free number in the interval

$$(x, x + x^{(1/(2k+w(k)))+\varepsilon}], \quad \text{where} \quad w(k) = 1/((k+1)2^k - 2k).$$

This improvement was so small that he decided to put the result aside for a couple years before returning to work on the subject further. Indeed, the use of exponential sum techniques already gave a small improvement of a similar nature. For $k = 2$, the above only leads to an improvement from $1/4$ in the exponent in Theorem 1.1 to $8/33 = 0.2424\ldots$. K. F. Roth [74] in his original paper on the subject already showed that exponential sums could replace $1/4$ with $3/13 = 0.2307\ldots$.

Early results in the literature for values of h such that the interval $(x, x+h]$ contains a squarefree number for every sufficiently large x include the following:

$h = x^{(2/5)+\varepsilon}$	E. Fogels [34], 1941 (elementary argument)
$h = x^{(1/4)+\varepsilon}$	K. F. Roth [74], 1951 (elementary argument)
$h = x^{3/13}(\log x)^{4/13}$	K. F. Roth [74], 1951
$h = c\,x^{1/4}$	M. Nair [68], 1979 (elementary argument)
$h = c\,x^{2/9}$	H. E. Richert [73], 1954
$h = x^{0.221982\ldots}$	R. A. Rankin [71], 1955
$h = x^{0.221585\ldots}$	P. G. Schmidt [77], 1964
$h = x^{0.2208986\ldots}$	S. W. Graham and G. Kolesnik [35], 1987.

In 1988, the first author [29] obtained the first elementary argument that produced an improvement on Theorm 1.1 that was not exponentially small. He showed that there is a constant c such that for x sufficiently large, there is a k-free number in the interval $(x, x + cx^{5/(10k+1)}]$. Around this time, V. Popov (for whom there is named a prize in the area of Approximation Theory) attended seminars that the first author gave on this result and returned home to the Bulgarian Academy of Sciences where he passed on material of the first author to the third author, a student of V. Popov. Thus began the collaboration of these two authors. This led to the following result [33] on gaps between squarefree numbers, based on elementary extensions of [74] and [38].

Theorem 3.1 (M. Filaseta and O. Trifonov, 1991). *There is a constant c such that for every x sufficiently large, there is a squarefree number in the interval $(x, x + c\,x^{1/5}\log x]$.*

Shortly after this, O. Trifonov [88] gave a complete extension of Theorem 1.1.

Theorem 3.2 (O. Trifonov, 1995). *Let k be an integer ≥ 2. There is a constant $c = c(k)$ such that for every x sufficiently large, there is a k-free number in the interval $(x, x + c\,x^{1/(2k+1)}\log x]$.*

Asymptotics for the number of k-free numbers in an interval $(x, x+h]$ also follow from these techniques if $h = cx^{1/(2k+1)}(\log x)w(x)$ where $w(x)$ is an arbitrary function of x that tends to infinity with x (see also [31]).

4. Integer points close to a curve

The above led to a general approach of using finite differences to aid in obtaining results that can be translated into a problem of bounding from above the number of integer points not necessarily on but close to a curve. The connection to Theorem 1.1 is transparent in the early discussions in this paper where we saw how differences and then modified differences, corresponding to the use of Padé approximants, were used to bound the number of integers $u \in (N, 2N]$ satisfying (1.1). Taking $f(u) = x/u^k$, we see that (1.1) only occurs for $u \in \mathbb{Z}$ for which there is a lattice point (u, m) that is $\le h/u^k$ from the curve $y = f(u)$. Thus, the approach in [38] can be viewed as introducing the use of differences to bound the number of lattice points close to a curve. Notably, another early paper that used differences, specifically divided differences, for the related problem of bounding the number of lattice points on a curve is the work of H. P. F. Swinnerton-Dyer [84]. This combination of using Padé approximants and using divided differences has continued to play an important role on advances in this area.

Among the numerous results that have ensued in this topic as a consequence of [38] are [31], [46,49], [51] and [87] (see also [47]). We give only a couple examples of results of this general nature. A result of this type that leads to Theorem 3.2 and can be found in [31] is the following.

Theorem 4.1 (M. Filaseta and O. Trifonov, 1996). *Let k be an integer ≥ 2, and let $s \in \mathbb{Q} - \{-(k-1), -(k-2), \dots, k-2, k-1\}$. Let δ and N be positive real numbers. Let $f(u) = X/u^s$, where X is an arbitrary real number independent of u and δ but possibly depending on k, N, and s. Suppose that*

$$N^s \le X \quad \text{and} \quad \delta \le c_k N^{-(k-1)}$$

where $c_k = c_k(k, s) > 0$ is sufficiently small. Set

$$S = \{u \in \mathbb{Z} \cap (N, 2N] : ||f(u)|| < \delta\}.$$

Then

$$|S| \ll_{k,s} X^{1/(2k+1)}\, N^{(k-s)/(2k+1)} + \delta\, X^{1/(6k+3)}\, N^{(6k^2+2k-s-1)/(6k+3)}.$$

The following result, further illustrating applications of using differences and which we will make use of in the next section, is from [51].

Theorem 4.2 (M. N. Huxley and P. Sargos, 2006). *Let $N \ge 4$, $\delta \le 1/4$, and $m \ge 3$. Suppose $f \in C^m$ with $|f^{(m)}(x)| \asymp \lambda_m$ and $|f^{(m-1)}(x)| \asymp \lambda_{m-1} = M\lambda_m$ for $N < x \le 2N$. Suppose further $\delta \ll \lambda_{m-1}$. Set*

$$S = \{u \in \mathbb{Z} \cap (N, 2N] : ||f(u)|| < \delta\}.$$

Then

$$|S| \ll N\, \lambda_m^{2/(m^2+m)} + N \left(\delta\, \lambda_m^{1/3}\right)^{2/(m^2-m+2)} + N\, \delta^{4/(m^2-3m+6)} + 1.$$

5. An example related to gaps between squarefree numbers

In this section, we show how the theorems in the previous section can be used to address a question posed by C. Spiro in the early 1990's at the West Coast Number Theory Conference in connection to her work [83] on the number of finite non-isomorphic groups of a given order. For n a positive integer, let p be a prime as small as possible with $n+p$ squarefree. She asked what kind of upper bound can be found for p. As an answer to this question has not appeared in the literature, we demonstrate here how results from the previous section can be used to address problems of this nature by establishing the following.

Theorem 5.1. *Let k be an integer ≥ 2. Then*

(i) There exist effectively computable positive constants C_k and N_k such that for each $n \geq N_k$ and each $h > C_k n^{1/(2k+1)} \log^2 n$, at least one-fifth of the primes $p \leq C_k n^{1/(2k+1)} \log^2 n$ are such that $n+p$ is k-free.

(ii) Let $h(n)$ be such that

$$\frac{h(n)}{n^{1/(2k+1)} \log^2 n} \to \infty \qquad \text{as } n \to \infty.$$

Then the number of primes $p \leq h(n)$ such that $n+p$ is k-free is

$$\prod_{\substack{q \nmid n \\ q \text{ prime}}} \left(1 - \frac{1}{q^{k-1}(q-1)}\right) \pi\big(h(n)\big)\,(1+o(1)).$$

We will be using repeatedly the following lemma which is standard in the literature.

Lemma 5.2. *Let P be a sequence of positive integers. Denote by $S(C,D)$ the number of elements of P which are in the interval $(C,D]$. Let $1 \leq A < B$, α, β, s,t, and u be positive real numbers such that*

$$S(M,2M) \ll sM^{\alpha} + t + uM^{-\beta} \tag{5.1}$$

for every $M \in (A, B/2]$ where the constant in $\ll$ depends only on α and β. Then $S(A,B) \ll sB^{\alpha} + t(\log(B/A)+1) + uA^{-\beta}$, where again the implied constant depends only on α and β.

Before beginning the proof, we set some notation. For $I = (C,D]$, we will sometimes use $S(I)$ to denote $S(C,D)$.

Proof. Let $k = [\log_2(B/A)]$. Hence, $2^k A \leq B < 2^{k+1}A$. Consider the intervals $I_j = (A2^j, A2^{j+1}]$ for $j \in \{0,1,\ldots,k-1\}$, and $I_k = (B/2, B] \subseteq (A2^k, B]$. Using (5.1) for each of these intervals, we get $S(I_j) \ll s(A2^j)^{\alpha} + t + u(A2^j)^{-\beta}$. Summing with respect to j we obtain

$$S(A,B) \ll sA^{\alpha} \sum_{j=0}^{k} 2^{\alpha j} + t(k+1) + uA^{-\beta} \sum_{j=0}^{k} 2^{-\beta j} \ll sA^{\alpha} 2^{k\alpha} + t(k+1) + uA^{-\beta}.$$

Since $2^k \le B/A$, this concludes the proof of the lemma. $\square$

Let $k \ge 2$ be a fixed integer. In particular, implied constants below may depend on k. Let $h > n^{1/(2k+1)}$ be as in the theorem. Our strategy will be to estimate $N = N(n,h)$ which we define as the number of primes $p \le h$ such that $n+p$ is not k-free. For such primes, $q^k | (n+p)$ for some prime q. In other words, $p \equiv -n \pmod{q^k}$ for some prime q. The number of primes $p \le h$ which are $\equiv -n \pmod{q^k}$ is $\pi(h; q^k, -n)$. Observe that $\pi(h; q^k, -n) = 0$ if $q^k > n + h$. Hence,

$$N \le \sum_{q \text{ prime}} \pi(h; q^k, -n).$$

For part (i) of Theorem 5.1, we will use the above bound on N. For part (ii), we will make use of a modified bound on N that makes more explicit use of its definition to handle the contribution from small primes q. Specifically, we use

$$N \le \sum_{p \le h} \chi_p + \sum_{\substack{q|n \text{ or } q > \log\log h \\ q \text{ prime}}} \pi(h; q^k, -n).$$

where

$$\chi_p = \begin{cases} 1 & \text{if } \exists \text{ a prime } q \le \log\log h \text{ with } q \nmid n \text{ such that } p \equiv -n \pmod{q^k} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\pi(h; q^k, -n) = 0$ or 1 if $q|n$, so the contribution to N from primes $q|n$ is at most $\omega(n)$, the number of distinct prime divisors of n. We easily have that $\omega(n) = o(\pi(h))$ since $h > n^{1/(2k+1)}$.

Next, we consider contribution of primes q not dividing n. For these, we note that $\pi(h; q^k, -n) \le 1$ whenever $q^k > h$.

Case 1. $q > h \log^6 n$.

Denote the contribution to N from primes $q > h \log^6 n$ by N_1. As noted above, $\pi(h; q^k, -n) \le 1$ in this case. Without loss of generality, we can assume $h < n$ since otherwise $q^k > n+h$. As before (see the derivation of (1.1)), we get that if $q^k|(n+p)$ for some $p \le h$, then $\|n/q^k\| < h/q^k$. There will be no advantage here in restricting ourselves to primes q, so we will estimate the number of integers u in the interval $(h \log^6 h, (n+h)^{1/k}]$ such that $\|n/u^k\| < h/u^k$.

We consider $(M, 2M] \subseteq (h \log^6 n, (n+h)^{1/k}]$ with an eye toward using Lemma 5.2. To estimate the number of integers $u \in (M, 2M]$ such that $\|n/u^k\| < h/u^k$, we use Theorem 4.1. Denote this number by $S_1(M)$. We take $X = n$, $s = k$, and $\delta = h/M^k$ in Theorem 4.1 to obtain

$$S_1(M) \ll n^{1/(2k+1)} + hM^{-1/3} n^{1/(6k+3)}.$$

Applying Lemma 5.2, we get

$$N_1 \ll n^{1/(2k+1)} \log n + h^{2/3} (\log n)^{-2} n^{1/(6k+3)} \ll n^{1/(2k+1)} \log n + h/(\log^2 n)$$

where we have used $h \ge n^{1/(2k+1)} \log^2 n$ for the last estimate.

Case 2. $q \in I_2 = (h/\log^2 n, M_1]$, where $M_1 = \min\{h \log^6 n, (n+h)^{1/k}\}$.

Denote the contribution to N from primes $q \in I_2$ by N_2. As in Case 1, we have $\pi(h; q^k, -n) \le 1$. Also, as before, we don't restrict ourselves to primes q and consider all integers $u \in I_2$, such that $u^k | (n+p)$ for some $p \le h$. Also, as before, we suppose as we may that $h < n$ since otherwise $q^k > n + h$. We consider $(M, 2M] \subseteq [h/\log^2 h, h \log^6 h]$ and denote the number of integers $u \in (M, 2M]$ satisfying $u^k | (n+p)$ for some $p \le h$ by $N_2(M)$. Again, for such u we have

$$\left\| \frac{n}{u^k} \right\| < \frac{h}{u^k}. \tag{5.2}$$

Let t be the positive integer such that

$$(h \log^6 n)^{t-1} \le \frac{n}{h} < (h \log^6 n)^t. \tag{5.3}$$

First, assume that $t \ge 3$. We take $f(u) = n/u^k$, $m = t$, $N = M$, $\delta = h/M^k$, $\lambda_{t-1} = n/M^{k+t-1}$ and $\lambda_t = n/M^{k+t}$ in Theorem 4.2. Note that (5.3) and $M \le h \log^6 n$ imply that $\delta \le \lambda_{t-1}$. Applying Theorem 4.2, we obtain

$$N_2(M) \ll M \lambda_t^{2/(t^2+t)} + M \big(\delta (\lambda_t)^{1/3}\big)^{2/(t^2-t+2)} + M \delta^{4/(t^2-3t+6)} + 1.$$

From (5.3) and $M \ge h/\log^2 n$, we have $\lambda_t/\delta = n/(hM^t) \le \log^{8t} n$. Thus,

$$N_2(M) \ll M \Big(\delta^{2/(t^2+t)} (\log n)^{16/(t+1)} + \delta^{8/(3(t^2-t+2))} (\log n)^{16t/(3(t^2-t+2))} + \delta^{4/(t^2-3t+6} \Big) + 1.$$

Since $t \ge 3$, we have $16/(t+1) \le 4$, $16t/(3(t^2-t+2)) \le 2$, and $2/(t^2+t) < 8/(3(t^2-t+2)) < 4/(t^2-3t+6)$. Therefore,

$$N_2(M) \ll M \left(\log^4 n \right) \delta^{2/(t^2+t)}.$$

Since $h > n^{1/(2k+1)}$, we have $t \le 2k$. Hence,

$$N_2(M) \ll M \left(\log^4 n \right) \delta^{1/(2k^2+k)} = M^{2k/(2k+1)} \left(\log^4 n \right) h^{1/(2k^2+k)}.$$

Applying Lemma 5.2, we deduce

$$N_2 \ll h^{(2k^2+1)/(2k^2+k)} \log^{10} n = h \frac{\log^{10} n}{h^{(k-1)/(2k^2+k)}} \ll h \frac{\log^{10} n}{n^{(k-1)/((2k^2+k)(2k+1))}},$$

since $h > n^{1/(2k+1)}$. Therefore, if n is sufficiently large, then

$$N_2 \ll \frac{h}{\log^2 n}.$$

We are left with considering $t \le 2$. Note that $t \le 2$ implies $h > n^{1/3}/\log^4 n$. We make use of an upper bound for the number of $u \in (M, 2M] \subseteq I_2$ for which u^k divides some integer in $(n, n+h]$. We apply Theorem 4.1 after making an adjustment on the interval $(n, n+h]$. We subdivide this interval into $\le (\log n)^3$ subintervals $(n', n'+h']$ with $h' = h/(\log n)^3$. We are interested now in an upper bound for the number of $u \in (M, 2M]$ for which u^k divides some integer in $(n', n'+h']$. We obtain here that such u satisfy $\|n'/u^k\| < h'/M^k$. We set $X = n'$, $s = k$, and $\delta = h'/M^k$ in Theorem 4.1. Observe that $M \geq h/(\log n)^2 = h' \log n$; hence, the condition $\delta < c_k/M^{k-1}$ holds for n large. Since $(M, 2M] \subseteq I_2$, we have $M \leq (n+h)^{1/k}/2$ so that $M^k \leq (n+h)/2^k < n < n'$. Thus, the conditions of Theorem 4.1 hold. We obtain that

$$\left|\{u \in (M, 2M] : \|n'/u^k\| < h'/M^k\}\right| \ll (n')^{1/(2k+1)} + h'M^{-1/3}(n')^{1/(6k+3)}.$$

Using $n' \leq n + h \leq 2n$ and $h' = h/(\log n)^3$, and taking into account that we have $(\log n)^3$ intervals $(n', n'+h']$, we deduce

$$N_2(M) \ll n^{1/(2k+1)}(\log n)^3 + hM^{-1/3}n^{1/(6k+3)}.$$

Since $k \geq 2$ and $M \geq h/\log^2 n$, Lemma 5.2 implies

$$N_2(M) \ll n^{1/5}(\log n)^4 + h^{2/3}(\log n)^{2/3}n^{1/15}.$$

Recalling that $t \leq 2$ gave us $h > n^{1/3}/\log^4 n$, we easily deduce

$$N_2 \ll \frac{h}{\log^2 n}.$$

Case 3. $q \in I_3 = (\log^2 n, h/\log^2 n]$.

Let N_3 be the contribution to N from primes $q \in I_3$. Since $\pi(h; q^k, -n) \leq (h/q^k) + 1$, we have

$$N_3 \leq \sum_{\log^2 n \leq q \leq h/\log^2 n} \left(\frac{h}{q^k} + 1\right) \ll \frac{h}{\log^2 n},$$

where we have used that

$$\sum_{q \geq \log^2 n} \frac{1}{q^k} \leq \sum_{q \geq \log^2 n} \frac{1}{q^2} \ll \frac{1}{\log^2 n}.$$

What remains is to deal with multiples of q^k when q is a prime $\leq \log^2 n$. First, we give estimates for proving part (i) of the theorem.

Case 4. $q \in I_4 = [11, \log^2 n]$.

Let N_4 be the contribution to N from $q \in I_4$. We use a version of Brun-Titchmarsh's inequality due to H. L. Montgomery and R. C. Vaughan [66].

Theorem 5.3 (H. L. Montgomery and R. C. Vaughan, 1973). *Let $m > 0$ and ℓ be integers with $(m, \ell) = 1$, and let $x > m$ be a real number. Then*

$$\pi(x; m, \ell) \leq \frac{2x}{\varphi(m)\log(x/m)}.$$

Recall that $h > n^{1/(2k+1)}$ and $q \leq \log^2 n$. We apply the above theorem with $x = h$, $m = q^k$, and $\ell = -n$ to get

$$\pi(h; q^k, -n) \leq \frac{2h}{q^{k-1}(q-1)\log(h/(\log^{2k} n))} \leq \frac{2h}{q(q-1)\log h}(1 + o(1)).$$

Thus, $N_4 \leq C_1 \pi(h)(1+o(1))$, where

$$C_1 = \sum_{\substack{q \geq 11 \\ q \text{ prime}}} \frac{1}{q(q-1)}.$$

We estimate this sum directly for primes $q \leq 5000$, and for $q > 5000$ we replace the sum with the corresponding sum over all integers > 5000 and rewrite it as a telescoping series to deduce that $C_1 < 0.0329$. Thus, for n is sufficiently large, $N_4 \leq 0.033\,\pi(h)$.

Case 5. $q \in \{2, 3, 5, 7\}$.

Let N_5 be the contribution to N from $q \in \{2, 3, 5, 7\}$. In this case, we use a result of O. Ramaré and R. Rumely [70].

Theorem 5.4 (O. Ramaré and R. Rumely, 1996). *Let $m \leq 72$ and ℓ be positive integers with $(m, \ell) = 1$, and let $x \geq 10^{10}$ be a real number. Then*

$$\left| \theta(x; m, \ell) - \frac{x}{\varphi(m)} \right| < .023269 \frac{x}{\varphi(m)}$$

and

$$|\theta(x) - x| \leq 0.000213x.$$

Note that, for every $\varepsilon \in (0, 1)$,

$$\theta(x; m, \ell) \geq \theta(x; m, \ell) - \theta(x^{1-\varepsilon}; m, \ell) \geq (1-\varepsilon)(\log x)\big(\pi(x; m, \ell) - \pi(x^{1-\varepsilon}; m, \ell)\big).$$

Thus,

$$\pi(x; m, \ell) \leq \frac{\theta(x; m, \ell)}{(1-\varepsilon)\log x} + x^{1-\varepsilon}.$$

In particular, for x sufficiently large, $\pi(x; m, \ell) < 1.01\,\theta(x; m, \ell)/\log x$. We get $\pi(h; m, \ell) < 1.034\,h/(\varphi(m)\log h)$ for h sufficiently large and m and ℓ satisfying the conditions of Theorem 5.4. So, for n sufficiently large, $N_5 \leq C_2 h/\log h$ where

$$C_2 = 1.034\left(\frac{1}{\varphi(4)} + \frac{1}{\varphi(9)} + \frac{1}{\varphi(25)} + \frac{1}{\varphi(49)}\right) = 1.034\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{20} + \frac{1}{42}\right) = 0.7656\ldots.$$

We are ready to prove part (i) of the theorem. We have

$$N \leq N_1 + N_2 + N_3 + N_4 + N_5.$$

Now, $N_1 + N_2 + N_3 \ll n^{1/(2k+1)} \log n + h/(\log^2 n)$. Since $h > C_k\, n^{1/(2k+1)} \log^2 n$, we have $N_1 + N_2 + N_3 < 0.001\,\pi(h)$ if C_k is sufficiently large. Also, $N_4 + N_5 \leq 0.033\,\pi(h) + 0.7657\,h/\log h$.

From the definition of $\theta(x)$, we obtain $\theta(x) \leq \pi(x)\log x$ for every $x \geq 1$. Using Theorem 5.4, for n and, hence, h sufficiently large, we obtain

$$\frac{h}{\log h} \leq \frac{\theta(h)}{0.999787\,\log h} < 1.00022\,\pi(h).$$

So, $N_4 + N_5 \le 0.799\,\pi(h)$. We deduce that $N \le 0.8\pi(h)$ for n and C_k sufficiently large, which proves part (i) of the theorem.

To prove part (ii) of the theorem, we estimate differently the contribution from $q \le \log^2 n$.

Case 6. $q \in I_6 = [\log\log h, \log^2 n]$.

Let N_6 be the contribution to N from $q \in I_6$. As in Case 4, we use Brun-Titchmarsh's inequality [66] to deduce

$$\begin{aligned} N_6 &\le \sum_{q \ge \log\log h} \frac{2h}{q(q-1)\log\left(h/(\log^{2k} n)\right)} \\ &\le \frac{2h}{(\log h)(\log\log h)}(1+o(1)) \ll \frac{\pi(h)}{\log\log h}. \end{aligned}$$

Case 7. $q \in I_7 = [2, \log\log h]$.

Let N_7 be the contribution to N from q in the above range. Thus, $N_7 \le \sum_{p\le h} \chi_p$, where χ_p is as defined before Case 1. For this case, we use the Siegel-Walfisz theorem (see [57]).

Theorem 5.5 (Siegel-Walfisz, 1936). *Let $A > 0$ be any fixed constant. Then there exists a positive constant $B = B(A)$ depending on A such that*

$$\pi(x; b, a) = \frac{\pi(x)}{\varphi(b)} + O\left(\frac{x}{\exp\left(B\sqrt{\log x}\right)}\right)$$

holds for sufficiently large values of x uniformly for $1 \le a < b$ with $\gcd(a, b) = 1$ and $b < (\log x)^A$.

Let the primes up to $\log\log h$ which do not divide n be $q_1, q_2, \ldots, q_t$, and let $P = q_1 q_2 \cdots q_t$. We will apply the Siegel-Walfisz theorem with $A = 2k$, $a = -n$ and $b = d^k$ where $d|P$. Observe that

$$P \le \prod_{\substack{q \le \log\log h \\ q \text{ prime}}} q \le \exp((1+o(1))\log\log h) < (\log h)^2.$$

Thus, $d|P$ implies $d^k < (\log h)^A$. Therefore, from Theorem 5.5, we have

$$\pi(h; d^k, -n) = \frac{\pi(h)}{\varphi(d^k)} + O\left(\frac{\pi(h)}{\log^2 h}\right),$$

where we emphasize that the implied constant depends only on k (with h sufficiently large). We apply the sieve of Eratosthenes now to obtain

$$\begin{aligned}\pi(h) - N_7 &= \sum_{d|P} \left(\frac{\mu(d)\pi(h)}{\varphi(d^k)} + O\left(\frac{\pi(h)}{\log^2 h}\right)\right) \\ &= \pi(h) \prod_{\substack{q \text{ prime}, q\nmid n \\ q \le \log\log h}} \left(1 - \frac{1}{q^{k-1}(q-1)}\right) + O\left(\frac{2^t \pi(h)}{\log^2 h}\right) \\ &= C\,\pi(h)(1+o(1)),\end{aligned}$$

where

$$C = \prod_{\substack{q\nmid n \\ q \text{ prime}}} \left(1 - \frac{1}{q^{k-1}(q-1)}\right).$$

Recall that $N_1 + N_2 + N_3 \ll n^{1/(2k+1)} \log n + h/(\log^2 n)$. Since $n^{1/(2k+1)} \log n = o(\pi(h))$, and $N_6 = o(\pi(h))$, part (ii) of the theorem follows.

6. A variety of applications

We have already illustrated a number of ways that Theorem 1.1 and the arguments given by H. Halberstam and K. F. Roth [38] have extended to produce other results of interest. Following the material of Section 4, a variety of applications into other somewhat different problems have arisen. In this section, we describe such applications. We do not give details of what goes into the proofs of these results, but rather rely on the previous section as an example for the general common theme in their arguments. In some cases, as with the earlier work mentioned on k-free values of polynomials, the role of the methods introduced by H. Halberstam and K. F. Roth play a historical boost into what was known on the topic and not necessarily the best or final word on the topic. In such cases, we also give the relevant references where improvements have been obtained.

6.1. *Squarefull numbers in short intervals*

A squarefull number is a positive integer n with the property that if $p|n$, then $p^2|n$. Let $Q(x)$ denote the number of squarefull numbers $\le x$. Then P. Bateman and E. Grosswald [2] showed that

$$Q(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O\big(x^{1/6} \exp(-c\,(\log x)^{4/7} (\log\log x)^{-3/7})\big)$$

for some constant $c > 0$. This asymptotic estimate implies that

$$Q\big(x + x^{(1/2)+\theta}\big) - Q(x) \sim \frac{\zeta(3/2)}{2\zeta(3)} x^{\theta} \quad \text{for } 1/6 < \theta < 1/2.$$

A number of results have been obtained improving on this lower bound for θ. This includes showing that one can obtain the following:

$\theta > 0.1526\ldots$	P. Shiu [80], 1984
$\theta > 0.1507\ldots$	P. G. Schmidt [76], 1986
$\theta > 0.1490\ldots$	C.-H. Jia [54], 1987
$\theta > 0.1425\ldots$	H. Liu [63], 1990
$\theta > 0.1318\ldots$	D. R. Heath-Brown [41], 1991
$\theta > 0.1308\ldots$	H. Liu [62], 1993
$\theta > 0.1282\ldots$	M. Filaseta and O. Trifonov [32], 1994
$\theta > 0.1250\ldots$	M. N. Huxley and O. Trifonov [52], 1996
$\theta > 0.1233\ldots$	O. Trifonov [87], 2002

The results since D. R. Heath-Brown's work [41] have all made use of an important idea of his. The last three papers [32], [52] and [87] all made use of further ideas similar to those given in Section 4. We also note that the use of divided differences similar to the work in H. P. F. Swinnerton-Dyer [84] played a crucial role in this topic. The best result to date on the topic [87], more explicitly, is given by $\theta > 19/154$.

6.2. *Moments of gaps between squarefree numbers*

Let $s_1, s_2, \ldots$ denote the squarefree numbers in ascending order. The problem here, introduced by P. Erdős [23], is to determine for which $\gamma \geq 0$ does there exist a constant $C(\gamma)$ such that

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^{\gamma} \sim C(\gamma)\, x. \tag{6.1}$$

Published results on this topic establish (6.1) holds for γ as follows:

$0 \leq \gamma \leq 2$	P. Erdős [23], 1951
$0 \leq \gamma \leq 3$	C. Hooley [44], 1973
$0 \leq \gamma \leq 3.22\ldots$	M. Filaseta [27], 1993
$0 \leq \gamma \leq 3.66\ldots$	M. N. Huxley [48], 1995

In 1984, C. Hooley announced (unpublished) an intermediate result in the list above, with his approach giving $0 \leq \gamma \leq 3.16\ldots$ is admissible in (6.1). The methods in [27] used a difference approach similar to the arguments leading to the results in Section 4. However, the argument of M. N. Huxley [48], giving (6.1) more precisely for $0 \leq \gamma \leq 11/3$, was instead based on an elegant use of the geometry of numbers.

6.3. *Gap result for the number of non-isomorphic abelian groups of a given order*

Let $a(n)$ be the number of non-isomorphic abelian groups of order n. For $k \in \mathbb{Z}^+$, let $A_k = \{n \in \mathbb{Z}^+ : a(n) = k\}$. The application here involves determining h as small as possible for which there exists a constant P_k such that the interval $(x, x+h]$ contains $P_k h + o(h)$ elements of A_k. The history here is shorter and can be summarized with the following admissible values for h:

$h = x^{0.33314...}$	E. Krätzel [60], 1980/A. Ivić [53], 1981
$h = x^{(1/5)+\varepsilon}$	H. Li [61], 1995
$h = x^{1/5}(\log x)g(x)$	M. Filaseta and O. Trifonov [31], 1996

In this last result, $g(x)$ is an arbitrary real valued function that satisfies $g(x) \to \infty$ as $x \to \infty$. E. Krätzel's [60,59] gave slightly stronger results than his listed above for certain residue classes of k modulo 30, the best of which was $h = x^{0.222...}$ when $k \equiv \pm 1 \pmod 6$. H. Li [61] made some use of exponential sums but also made use of [33] and, hence, the idea of using a difference approach as discussed previously in this paper. The first and third author in [31] noted that one could bypass the use of exponential sums in H. Li's argument and obtain a result that is analogous to what can be obtained for gaps between squarefree numbers.

6.4. *Binomial coefficients with all large prime divisors*

In 1974, E. F. Ecklund, Jr., P. Erdős and J. L. Selfridge considered the problem of determining, for a fixed $k \in \mathbb{Z}$ with $k \geq 2$, a lower bound for the least integer $g(k) > k+1$ for which all prime divisors of $\binom{g(k)}{k}$ are $> k$. The following estimates have been obtained.

$g(k) > k^{1+c}$	E. F. Ecklund, Jr., P. Erdős and J. L. Selfridge [21], 1974
$g(k) > ck^2/\log k$	P. Erdős, C. B. Lacampagne and J. L. Selfridge [24], 1993
$g(k) > \exp\left(\dfrac{c\,(\log k)^{3/2}}{(\log\log k)^{1/2}}\right)$	A. Granville and O. Ramaré [36], 1996
$g(k) > \exp\left(c\log^2 k\right)$	S. Konyagin [58], 1999

In each case c is a positive constant, which if sufficiently small can be taken to be the same. S. Konyagin [58] introduced some new ideas using differences and the geometry of numbers to obtain estimates of the type mentioned in Section 4 and then applied them to obtain his result.

6.5. *Prime powers with a special property*

In connection to work of J. P. Serre [78,79] on $\mathbb{F}_q$-rational points on curves of small genus over the finite field $\mathbb{F}_q$, F. Luca and I. E. Shparlinski [64] in 2009 considered the problem of bounding $N(Q)$, where $Q \geq 8$ and $N(Q)$ is the number of prime powers $q \leq Q$ of the form $q = p^{2k+1}$ with p a prime and $k \geq 1$ an integer and with p dividing $\lfloor 2\sqrt{q} \rfloor$. They showed that there exist positive constants c_1 and c_2 such that

$$c_1(\log Q)^{1/2} \leq N(Q) \leq c_2 Q^{17/140} = c_2 Q^{0.1214\ldots}.$$

D. Baczkowski and O. Trifonov (in unpublished work, see [1]) have shown one can take the exponent in the upper bound to be $5/42 = 0.1190\ldots$. For these upper bounds on $N(Q)$, estimates of the type given in Section 4 were used.

6.6. *An estimate involving Stirling numbers of the second kind*

The Stirling number of the second kind, $S(n,k)$, is the number of ways of writing the set $\{1, 2, \ldots, n\}$ as a union of k non-empty pairwise disjoint subsets of $\{1, 2, \ldots, n\}$. It can be shown that either

(i) $\exists\, K_n$ such that $S(n, K_n) > S(n,k)$ for all $k \neq K_n$, or

(ii) $\exists\, K_n$ such that $S(n, K_n) = S(n, K_n + 1) > S(n,k)$ for all $k \notin \{K_n, K_n + 1\}$.

The n for which the latter hold appear to be exceptional (cf. [19]). Denote the set of them by E. E. R. Canfield and C. Pomerance [19] showed showed that

$$|\{n \leq x : n \in E\}| \ll x^{(3/5)+\varepsilon},$$

and G. Kemkes, D. Merlini and B. Richmond [56] improved this to

$$|\{n \leq x : n \in E\}| \ll x^{(1/2)+\varepsilon}.$$

E. R. Canfield and C. Pomerance [19] used estimates for integer points close to a curve similar to the results in Section 4, whereas G. Kemkes, D. Merlini and B. Richmond [56] used work of E. Bombieri and J. Pila [17] on estimates for integer points on a curve.

7. Conclusion

The work of K. F. Roth [74] and by H. Halberstam and K. F. Roth [38] which gave us Theorem 1.1 has made an impact on a number of related and not-so-related number theoretic problems. In addition, their work has led to the development of approaches using finite differences, to estimate in particular the number of integer points that are close to a curve, that have been fruitful in a number of these applications. As these implications of their papers have continued to progress with time, we can look forward to seeing the influence of their work in research for years to come.

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