FURTHER IRREDUCIBILITY CRITERIA FOR POLYNOMIALS WITH NON-NEGATIVE COEFFICIENTS

MORGAN COLE, SCOTT DUNN, AND MICHAEL FILASETA

Abstract. Let \( f(x) \) be a polynomial with non-negative integer coefficients. This paper produces sharp bounds \( M_1(b) \) depending on an integer \( b \in [3, 20] \) such that if each coefficient of \( f(x) \) is \( \leq M_1(b) \) and \( f(b) \) is prime, then \( f(x) \) is irreducible. A number of other related results are obtained.

1. Introduction

If \( d_n d_{n-1} \ldots d_1 d_0 \) is the decimal representation of a prime, then a result of A. Cohn in [11] asserts that

\[
f(x) = d_n x^n + d_{n-1} x^{n-1} + \cdots + d_1 x + d_0
\]

is irreducible over the integers. This paper is inspired by the following two natural questions. If one views \( f(x) \) as being a general polynomial with non-negative integer coefficients with \( f(10) \) prime, then does the irreducibility of \( f(x) \) in \( \mathbb{Z}[x] \) really depend on its coefficients being less than 10? Is there a particular reason that base 10 is special or do analogous results hold when 10 is replaced by some other integer?

Some answers to these questions have been given already in the literature. The result of Cohn has been extended to all bases \( b \geq 2 \) by J. Brillhart, A. Odlyzko and the third author [3], to base \( b \) representations of \( kp \) where \( k \) is a positive integer \( < b \) and \( p \) is a prime by the third author [5] (also see [8]), and to an analog in function fields over finite fields by R. Murty [9]. Furthermore, [3] allows the coefficients \( d_j \) in Cohn’s theorem to satisfy \( 0 \leq d_j \leq 167 \) rather than \( 0 \leq d_j \leq 9 \); and later the third author [6] showed that the coefficients \( d_j \) need only satisfy \( 0 \leq d_j \leq 10^{30} d_n \) and, further, that simply \( d_j \geq 0 \) suffices if \( n \leq 31 \). Some further work on upper bounds for \( d_j \) can be found in [1] and [2].

Recent work by S. Gross and the third author [7] extended this last line of investigation even further. They showed that if \( f(x) \) is a polynomial with

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non-negative coefficients bounded above by

\[ 49598666989151226098104244512918 \]

and \( f(10) \) is prime, then \( f(x) \) is irreducible over \( \mathbb{Z} \). They also showed that if instead the coefficients were bounded above by

\[ 8592444743529135815769545955936773, \]

then \( f(x) \) is either irreducible over \( \mathbb{Z} \) or divisible by \( x^2 - 20x + 101 \). Furthermore, and perhaps most surprising, they established that these two upper bounds are sharp.

The main goal of this paper is to extend the results in [7] to different bases. We focus on bases \( b \in [2, 20] \). As we will see, the smaller the base, the more difficult the analysis becomes. In the way of notation, we use \( \Phi_n(x) \) to denote the \( n \)-th cyclotomic polynomial, and irreducibility throughout will refer to irreducibility in \( \mathbb{Z}[x] \). Our main goal is to establish the following.

**Theorem 1.1.** Fix an integer \( b \in [2, 20] \), and let \( M_1(b) \) and \( M_2(b) \) be as given in Table 1 and Table 2, respectively. Let \( f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x] \) with \( a_j \geq 0 \) for each \( j \) and \( f(b) \) prime. If each \( a_j \leq M_1(b) \), then \( f(x) \) is irreducible. Also, for \( 3 \leq b \leq 5 \), if each \( a_j \leq M_2(b) \) and \( f(x) \) is reducible, then \( f(x) \) is divisible by \( \Phi_3(x - b) \). Similarly, for \( 6 \leq b \leq 20 \), if each \( a_j \leq M_2(b) \) and \( f(x) \) is reducible, then \( f(x) \) is divisible by \( \Phi_4(x - b) \).

We will show that, for \( 3 \leq b \leq 20 \), the bound \( M_1(b) \) is sharp. For \( 4 \leq b \leq 20 \), we will likewise show that the bound \( M_2(b) \) is sharp.

We suspect the bound \( M_1(2) = 7 \) as given in Table 1 is not sharp. Of some related interest is the example

\[ f(x) = x^{15} + 9x^{10} + 9x^9 + 9x^8 + 9x^7 + 9x^6 + 8x^5 + 10x^4 + 7x^3 + 10x^2 + 9x + 3. \]

Here \( f(2) = 51157 \) is prime, the largest coefficient of \( f(x) \) is 10, and \( f(x) \) is divisible by \( x^2 - 3x + 3 \). This example shows that the largest permissible value of \( M_1(2) \) is \( \leq 9 \). Therefore, this largest permissible value is 7, 8 or 9.

Computations in this paper were done using MAPLE 2015. The “is-prime” routine was used to detect likely primes in our computations, and these were verified by using primality tests in Sage Version 4.6.

2. Preliminary Results

We begin with an instructive lemma adapted from [3].

**Lemma 2.1.** Fix an integer \( b \geq 2 \). Let \( f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x] \) be such that each \( a_j \geq 0 \) and \( f(b) \) is prime. If \( f(x) \) is reducible, then \( f(x) \) has a non-real root in the disc \( \mathcal{D}_b = \{ z \in \mathbb{C} : |b - z| \leq 1 \} \).
<table>
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<th>$M_1(b)$</th>
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Table 1. $M_1(b)$ for $2 \leq b \leq 20$
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Table 2. $M_2(b)$ for $3 \leq b \leq 20$
Theorem 2.2. Fix an integer $b$ such that $b \geq 2$, and let $D = D(b)$ as given in Table 3. Let $f(x) = \sum_{j=0}^{n} a_j x^j$ be a non-constant polynomial in $\mathbb{Z}[x]$ with each $a_j \geq 0$ and with $f(b)$ prime. If the degree of $f(x)$ is $\leq D$, then $f(x)$ is irreducible.

<table>
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<td>50</td>
<td>53</td>
<td>56</td>
<td>59</td>
<td>62</td>
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</table>

Table 3. Maximum degree based on $b$

Proof. By way of contradiction, assume $f(x)$ is reducible. Then $f(x)$ has a non-real root $\alpha \in \mathcal{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}$ by Lemma 2.1. Since the complex conjugate of $\alpha$ is also a root of $f(x)$, we may assume that $\alpha$ has a positive imaginary part.

Note that the line passing through the origin and tangent to $\mathcal{D}_b$ from above has slope $\sin^{-1}(1/b)$. We write $\alpha = re^{i\theta}$, where $r \geq b - 1$ and $0 < \theta \leq \sin^{-1}(1/b)$. A direct computation shows that for each $k \in \{1, 2, \ldots, D\}$, we have that $0 < k\theta \leq D\sin^{-1}(1/b) < \pi$. This gives us that

$$\text{Im}(\alpha^k) = r^k \sin(k\theta) > 0 \text{ for } 1 \leq k \leq D.$$
Our polynomial \( f(x) \) has non-negative coefficients and \( \deg f = n \) with \( 1 \leq n \leq D \), so
\[
\text{Im}(f(\alpha)) \geq \text{Im}(\alpha^n) > 0,
\]
but this contradicts the fact that \( \alpha \) is a root of \( f(x) \). Thus, \( f(x) \) is irreducible. \( \square \)

The bounds \( D(b) \) given in Table 3 are not necessarily sharp, but are for many \( b \). Take for example \( b = 4 \). We see that
\[
f(x) = x^{13} + x^3 + 235835x + 16576651
\]
is of degree 13, \( f(4) = 84628919 \) is prime, each coefficient is \( \leq 16576651 \), and \( f(x) \) is divisible by \( \Phi_3(x-4) = x^2 - 7x + 13 \). Thus, \( D(4) \) in Table 3 is sharp.

In Section 4, we will give sharp bounds \( D(b) \) for all \( b \in [2, 20] \). Additionally, although not the focus of this paper, we will give sharp bounds on the size of the coefficients when \( f(x) \) is reducible and of degree \( D(b) + 1 \).

A motivating idea for our next two sections is to replace the disk \( D_b \) in Lemma 2.1 with a set of points such that if \( \alpha = re^{i\theta} \) is in the new set of points, then \( |\theta| \) is bounded above by a number smaller than \( \sin^{-1}(1/b) \). This then will allow us to determine sharp bounds for \( D(b) \) in place of those given in Table 3 for Theorem 2.2.

### 3. A Root Bounding Function

For a given \( b \in \{2, 3, \ldots, 20\} \), our main goal is to establish the upper bounds \( M_1(b) \) and \( M_2(b) \) given in Theorem 1.1 and, further, to show that they are sharp when they are sharp as described after the statement of Theorem 1.1. We will utilize three main methods as in [7]. First, we will introduce certain rational functions that will give us information on the location of possible roots of \( f(x) \). These rational functions will vary depending on \( b \). Even in the case \( b = 10 \), we will be able to obtain slightly better information than in [7] by using a modification of the rational function given there. Second, we obtain an initial value for \( M_1(b) \) and \( M_2(b) \) using a result first introduced in [1] and [2] but based on the main ideas in the earlier work [6]. Third, we use information gained from recursive relations on the possible factors of \( f(x) \), as outlined in [7], to establish sharp values of \( M_1(b) \) for \( b \geq 3 \) and sharp values of \( M_2(b) \) for \( b \geq 4 \). In this section, we focus on the first of these ideas.

We recall that \( \Phi_n(x) \) denotes the \( n \)-th cyclotomic polynomial, and we use \( \zeta_n = e^{2\pi i/n} \). Fix an integer \( b \) with \( 2 \leq b \leq 20 \). Let \( f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x] \) satisfying \( a_j \geq 0 \) and \( f(b) \) is prime.
As in the proof of Lemma 2.1, we consider the case that $f(x)$ is reducible so that $f(x) = g(x)h(x)$, where each of $g(x)$ and $h(x)$ are polynomials with integer coefficients and are not identically $\pm 1$. We may suppose further that $g(x)$ and $h(x)$ have positive leading coefficients, and do so. Given that $f(p)$ is prime, we take, without loss of generality, $g(b) = \pm 1$. Lemma 2.1 implies that $g(x)$ has a non-real root in the disc $D_b = \{ z \in \mathbb{C} : |b - z| \leq 1 \}$. Using the ideas of [7], we wish to show that either $g(x)$ has a root in common with one of $\Phi_3(x-b) = x^2 - (2b - 1)x + b^2 - b + 1$, $\Phi_4(x-b) = x^2 - 2bx + b^2 + 1$, and $\Phi_6(x-b) = x^2 - (2b + 1)x + b^2 + b + 1$, or $g(x)$ has roots in a certain region $R_b$ to be defined shortly.

We define

$$F_b(z) = \frac{N_b(z)}{D_b(z)},$$

where

$$N_b(z) = |b - 1 - z|^{2e_2} (|b + \zeta_3 - z| |b + \overline{\zeta_3} - z|)^{2e_3} \cdot (|b + i - z| |b - i - z|)^{2e_4} (|b + \zeta_6 - z| |b + \overline{\zeta_6} - z|)^{2e_6},$$

$$D_b(z) = |b - z|^{4(e_3 + e_4 + e_6) + 2(e_2 + d + 1)},$$

and $e_2 = e_2(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$ and $d = d(b)$ are all non-negative integers. For Theorem 1.1, the numbers $e_2, e_3, e_4, e_6$ and $d$ for a given $b$ are given in Table 4.

<table>
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<tr>
<th>$b$</th>
<th>$2$</th>
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<td>$e_6 = e_6(b)$</td>
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</table>

**Table 4.** Numbers used in $F_b(z)$ for $b$

We note that these are not the only choices for $e_2(b), e_3(b), e_4(b), e_6(b)$, and $d(b)$ that can serve our purposes. For example, the choice of $e_2(10) = 0$, $e_3(10) = 3$, $e_4(10) = 2$, $e_6(10) = 3$, and $d(10) = 3$ are the numbers for $b = 10$ that were used in [7]. Our choices for the numbers in Table 4 are based on trial and error to see what would give us the best results. In the
case of $b = 10$, there is a slight advantage that will arise from the use of the $e_j$’s given in Table 4.

Setting $z = x + iy$, it is not difficult to see or to use direct computations to verify that each of the expressions

$$|b - 1 - z|^2, \ (|b + \zeta_3 - z||b + \overline{\zeta_3} - z|)^2, \ (|b - i - z||b - i - z|)^2$$

$$(|b + \zeta_6 - z||b + \overline{\zeta_6} - z|)^2 \quad \text{and} \quad |b - z|^2$$

is a polynomial in $\mathbb{Z}[b,x,y]$. Therefore, $N_b(z)$ and $D_b(z)$ are polynomials in $\mathbb{Z}[b,x,y]$, so $F_b(z)$ is a rational function in $b$, $x$ and $y$.

We write $g(x)$ in the form

$$g(x) = c \prod_{j=1}^{r}(x - \beta_j),$$

where $c$ is the leading coefficient of $g(x)$ and $\beta_1, \ldots, \beta_r$ are the roots of $g(x)$, and therefore also roots of $f(x)$. For ease of notation, we define

$$\tilde{g}_b(n) = g(b \pm \zeta_n) g(b \pm \overline{\zeta_n}).$$

One then checks that the two expressions

$$\frac{|g(b - 1)|^{2e_2}|\tilde{g}_b(3)|^{2e_3}|\tilde{g}_b(4)|^{2e_4}|\tilde{g}_b(6)|^{2e_6}}{|g(b)|^{4(e_3 + e_4 + e_6) + 2(e_2 + d + 1)}}$$

and

$$\frac{1}{c^{2(d + 1)}} \prod_{j=1}^{r} F_b(\beta_j)$$

are equal. We denote this common value by $V = V_b(g)$.

Now, each of $\tilde{g}_b(3)$, $\tilde{g}_b(4)$ and $\tilde{g}_b(6)$ is a symmetric polynomial, with integer coefficients, in the roots of an irreducible monic quadratic in $\mathbb{Z}[x]$. Hence, each of these expressions is an integer. Also, $g(b - 1)$ is an integer. Thus, the numerator of the first expression for $V$ above is an integer. Since $g(b) = \pm 1$ and $V \geq 0$, we know that either $V = 0$ or $V \in \mathbb{Z}^+$. We recall that $f(x)$ is a polynomial with non-negative integer coefficients. Thus, $f(x)$ cannot have a positive real root, and neither can $g(x)$ which is a factor of $f(x)$. Therefore, $g(b - 1) \neq 0$. Either definition of $V$ now implies that $V = 0$ if and only if at least one of $\Phi_3(x - b)$, $\Phi_4(x - b)$ and $\Phi_6(x - b)$ is a factor of $g(x)$. If none of these quadratics is a factor of $g(x)$, we necessarily have that $V \in \mathbb{Z}^+$. In this case, the product in the second expression for $V$ above must be a positive integer. Since $F_b(z)$ is a non-negative real number for all $z \in \mathbb{C}$, we deduce that $F_b(\beta_j) \geq 1$ for at least one value of $j \in \{1,2,\ldots,r\}$. In other words, there is a root $\beta$ of $g(x)$, and consequently of $f(x)$, satisfying $F_b(\beta) \geq 1$. 
Summarizing the above ideas, given only that \(g(x) \in \mathbb{Z}[x], g(b - 1) \neq 0, g(x) \neq \pm 1\) and \(g(b) = \pm 1\), we have shown that either \(g(x)\) has at least one of the factors \(\Phi_3(x - b), \Phi_4(x - b)\) and \(\Phi_6(x - b)\), or \(g(x)\) has a root \(\beta\) in the region \(\mathcal{R}_b\) defined as

\[
\mathcal{R}_b = \{z \in \mathbb{C} : F_b(z) \geq 1\}.
\]

In the latter case, we will use an analysis of the region \(\mathcal{R}_b\) in the complex plane to obtain important information about the location of \(\beta\).

It is of some interest to note that the conditions above that \(g(x) \in \mathbb{Z}[x], g(x) \neq \pm 1\) and \(g(b) = \pm 1\), are sufficient to show that \(g(x)\) has a root in \(\mathcal{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}\). The following graphs depict regions \(\mathcal{R}_b\) for \(b \in \{2, 3, 4\}\) where \(e_2(b), e_3(b), e_4(b), e_6(b)\) and \(d(b)\) are as given in Table 4. The circle imposed on the graph is the unit circle centered at \(b\), the boundary of \(\mathcal{D}_b\). These graphs are, of course, obtained from plotting only a finite set of points and are not used in our proofs but are intended to help visualize \(\mathcal{R}_b\).

Figure 1. Image of \(\mathcal{R}_2\)

Figure 2. Image of \(\mathcal{R}_3\)

Figure 3. Image of \(\mathcal{R}_4\)

Figure 4 shows \(\mathcal{R}_{10}\) for our choice of \(e_2(10) = 0, e_3(10) = 4, e_4(10) = 2, e_6(10) = 3\) and \(d(10) = 3\) while Figure 5 shows \(\mathcal{R}_{10}\) for the choice of \(e_2(10) = 0, e_3(10) = 3, e_4(10) = 2, e_6(10) = 3\) and \(d(10) = 3\) used in [7].
Figure 4. Our choice for $\mathcal{R}_{10}$

Figure 5. $\mathcal{R}_{10}$ used in [7]

Although subtle, Figure 5 is symmetric about the vertical line $x = 10$, while Figure 4 is slightly narrower at the front of the region.

In what follows, we will sometimes refer to points $(x, y)$ in $\mathcal{R}_b$, and this is to be interpreted as the point $z = x + iy$ in the complex plane in $\mathcal{R}_b$. For example, taking $b = 6$, we will see later that all the points $(x, y) \in \mathcal{R}_6$ lie below the line $y = \tan(\pi/21)x$. This then means that any point $z = x + iy \in \mathcal{R}_6$ satisfies $y \leq \tan(\pi/21)x$.

To further help us analyze the region $\mathcal{R}_b$, we define

\begin{equation}
    P_b(x, y) = D_b(x + iy) - N_b(x + iy).
\end{equation}

Direct computations for each $b \in \{2, 3, \ldots, 20\}$ show that we can write

\begin{equation}
    P_b(x, y) = \sum_{j=0}^{r} a_j(b, x)y^{2j},
\end{equation}

where $r = 2(e_3 + e_4 + e_6) + e_2 + d + 1$ and each $a_j(b, x)$ is an integer polynomial in $b$ and $x$. Furthermore, the definition of $D_b(z)$ implies that $D_b(z) > 0$ for all $z \in \mathbb{C}$ with $z \neq b$. Thus,

\[ F_b(x + iy) \geq 1 \quad \text{and} \quad P_b(x, y) \leq 0 \]

are equivalent for $z \neq b$. Also, we have that the equation $F_b(x + iy) = 1$ and $P_b(x, y) = 0$ are equivalent for $z \neq b$. Note that $P_b(b, 0) = D_b(b) - N_b(b) = 0 - 1 = -1$. Therefore, the $z = x + iy \in \mathbb{C}$ such that $F_b(z) = 1$ correspond exactly to the points $(x, y)$ where $P_b(x, y) = 0$.

We introduce the following technical lemma that corresponds to Lemma 2 in [7].

**Lemma 3.1.** Fix an integer $2 \leq b \leq 20$. Then there exist real numbers $a_0 = a_0(b)$, $a_1 = a_1(b)$, and a non-negative real-valued function $\rho_b(x)$ defined on the interval $I_b = [b - a_0, b + a_1]$ such that the following conditions hold:

(i) For every $x \not\in I_b$ and every $y \in \mathbb{R}$, we have $P_b(x, y) \neq 0$.

(ii) For all $x \in I_b$, we have $P_b(x, \rho_b(x)) = 0$.

(iii) Both $\rho_b(b - a_0) = 0$ and $\rho_b(b + a_1) = 0$ hold.
(iv) The function $\rho_b(x)$ is a continuously differentiable function on the interior of $I_b$ and is continuous on $I_b$.

(v) If $x$ and $y$ are real numbers for which $P_b(x, y) \leq 0$, then $x \in I_b$ and $|y| \leq \rho_b(x)$.

Given the above lemma, complex numbers of the form $x + i\rho_b(x)$ are boundary points of $R_b$ which are on or above the real axis. Since $P_b(x, y)$ is a polynomial in $y^2$ with coefficients in $\mathbb{Z}[b, x]$, our region $R_b$ is symmetric about the real axis. Thus, the points $x - i\rho_b(x)$ are boundary points of $R_b$ which are on or below the real axis. The points $b-a_0$ and $b+a_1$ are boundary points on the real axis.

To prove Lemma 3.1, we use the Implicit Function Theorem (cf. [12]), which we state next.

**Lemma 3.2.** Let $\mathcal{D}$ be an open set in $\mathbb{R}^2$ and let $W : \mathcal{D} \to \mathbb{R}$. Suppose $W$ has continuous partial derivatives $W_x$ and $W_y$ on $\mathcal{D}$. Let $(x_0, y_0) \in \mathcal{D}$ be such that

$$W(x_0, y_0) = 0 \text{ and } W_y(x_0, y_0) \neq 0.$$ 

Then there is an open interval $I \subseteq \mathbb{R}$ and a real valued, continuously differentiable function $\phi$ defined on $I$ such that $x_0 \in I$, $\phi(x_0) = y_0$, $(x, \phi(x)) \in \mathcal{D}$ for all $x \in I$, and $W(x, \phi(x)) = 0$ for all $x \in I$.

Our proof of Lemma 3.1 is a variation of the proof given for Lemma 2 in [7]. A number of changes and some simplifications are introduced. In particular, the proof of Lemma 2 in [7] used more than once that a certain discriminant is non-zero which no longer applies in our case, so some changes in the arguments here become necessary.

We give a proof based on the values of $(e_2, e_3, e_4, e_6, d)$ given in Table 4 for each $b$. Before delving into the proof, we note that we will want analogous results for other choices of $(e_2, e_3, e_4, e_6, d)$ in the next section and that the same lemma holds following the same line of argument. Specifically, we will additionally use Lemma 3.1 for $(e_2, e_3, e_4, e_6, d) = (0, 2, 1, 0, 1)$ and $b = 2$, for $(e_2, e_3, e_4, e_6, d) = (0, 2, 3, 0, 8)$ and $b = 3$, for $(e_2, e_3, e_4, e_6, d) = (0, 2, 4, 0, 8)$ and $b = 4$ or 5, for $(e_2, e_3, e_4, e_6, d) = (0, 2, 5, 0, 12)$ and $b = 6$ or 7, for $(e_2, e_3, e_4, e_6, d) = (0, 1, 8, 0, 14)$ and $8 \leq b \leq 14$, and for $(e_2, e_3, e_4, e_6, d) = (0, 1, 10, 0, 24)$ and $15 \leq b \leq 20$.

**Proof of Lemma 3.1.** We fix an integer $b \in [2, 20]$, and let $e_2 = e_2(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$ and $d = d(b)$ be as in Table 4. We set $r = 2(e_3 + e_4 + e_6) + e_2 + d + 1$, and let $P_b(x, y)$ be as in (3.4). For $0 \leq j \leq r$,
define \( p_j(b, x) = a_j(b, x + b) \), and set
\[
\hat{P}_b(x, y) = \sum_{j=0}^r p_j(b, x)y^j = \sum_{j=0}^r a_j(b, x + b)y^j.
\]
Thus,
\[
(3.5) \quad \hat{P}_b(x, y^2) = P_b(x + b, y).
\]
Observe that the points \((x, y)\) corresponding to \( \hat{P}_b(x, y^2) \leq 0 \) are the points \((x - b, y)\) where \((x, y)\) \(\in\) \(\mathcal{R}_b\); in other words, the \((x, y)\) satisfying \( \hat{P}_b(x, y^2) \leq 0 \) correspond to the \((x, y) \in \mathcal{R}_b\) translated to the left by \(b\).

For fixed \(b \in [2, 20]\), the expression \(p_j\) is a polynomial with integer coefficients in the variable \(x\). The dependence on \(b\) only arises in our choice of \(e_2(b), e_3(b), e_4(b), e_6(b)\) and \(d(b)\). Since the same choice of \(e_2(b), e_3(b), e_4(b), e_6(b)\) and \(d(b)\) are used for each \(b \in [6, 20]\), we have only five sets of \(p_j(b, x)\) to consider. We computed these explicitly to help with the analysis that follows.

To simplify our notation and avoid confusion, we use \(\hat{P}_b(y)\) for \(\hat{P}_b(x, y)\) when we are viewing \(\hat{P}_b(x, y)\) as a polynomial in \(y\) whose coefficients are polynomials in \(x\). Table 5 lists \(r\), the degree of \(\hat{P}_b(y)\), for each \(b\).

<table>
<thead>
<tr>
<th>(b)</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>38</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>26</td>
</tr>
<tr>
<td>(6 \leq b \leq 20)</td>
<td>22</td>
</tr>
</tbody>
</table>

Table 5. Degree \(r\) of \(\hat{P}_b(y)\) for \(b\)

Using a Sturm sequence, we verify that \(p_0(b, x)\) has exactly two distinct real roots. One checks that \(p_0(b, x) = 0\) has a negative root, which we denote by \(-a_0\), and a positive root, which we will call \(a_1\). Computations give us the values of \(a_0\) and \(a_1\) for \(b \in [2, 20]\), accurate to the digits shown in Table 6. We show that \(a_0\) and \(a_1\) have the properties stated in Lemma 3.1.

<table>
<thead>
<tr>
<th>(b)</th>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(\hat{a}_0)</th>
<th>(\hat{a}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.5523770847</td>
<td>10.0651310946</td>
<td>0.5523</td>
<td>10.06</td>
</tr>
<tr>
<td>3</td>
<td>1.0721963435</td>
<td>3.4397713145</td>
<td>1.07</td>
<td>3.43</td>
</tr>
<tr>
<td>4</td>
<td>1.3782037799</td>
<td>2.4446162254</td>
<td>1.37</td>
<td>2.44</td>
</tr>
<tr>
<td>5</td>
<td>1.4754548441</td>
<td>2.0416766993</td>
<td>1.47</td>
<td>2.04</td>
</tr>
<tr>
<td>(6 \leq b \leq 20)</td>
<td>1.5638035689</td>
<td>1.7605007116</td>
<td>1.56</td>
<td>1.76</td>
</tr>
</tbody>
</table>

Table 6. Values of \(a_0\) and \(a_1\) for \(b \in [2, 20]\)
Let $J_b$ denote the interval $[-a_0, a_1]$. Using a Sturm sequence, one can verify that for each $j \in \{1, 2, \ldots, r\}$, the polynomial $p_j(b, x)$ has all of its real roots in the interval $[-\hat{a}_0, \hat{a}_1] \subset J_b$, where $\hat{a}_0$ and $\hat{a}_1$ are given in Table 6.

Recalling (3.5), we see that to prove part (i), we need only show that for each $x_0 \notin J_b$, the real roots of $P_b(x_0, y)$ are all negative. A simple calculation shows that $p_j(b, \pm 11) > 0$ for all $j \in \{0, 1, \ldots, r\}$ (and each $b$). Since each $p_j(b, x)$ has its real roots inside $J_b$, we deduce that $p_j(b, x_0) > 0$ for each $j$. From Descartes’ rule of signs, we obtain that $\tilde{P}_b(x_0, y)$ has no positive real roots. Part (i) now follows. We note for further use that we also have

\begin{equation}
P_b(x, y) > 0 \quad \text{for all } x \notin I_b \text{ and all } y \in \mathbb{R}.
\end{equation}

We turn to the remaining parts of Lemma 3.1. For a given $x \in I_b$, we want to define $\rho_b(x)$ as the largest non-negative real root of $P_b(x, y)$. First, however, we need to show that such a non-negative real root exists. From (3.5), we see that for $x \in J_b$, we want $(\rho_b(x + b))^2$ to be a root of $\tilde{P}_b(y)$. Further, showing $P_b(x, y)$ has a non-negative real root for each $x \in I_b$ is equivalent to showing $\tilde{P}_b(y)$ has a non-negative real root for each $x \in J_b$.

A direct computation gives that $p_0(b, 0) = -1$ and $p_r(b, x) = 1$. Since $p_j(b, x)$ has only the two real roots $-a_0$ and $a_1$, it follows that $p_0(b, x_0) < 0$ for all $x_0 \in (-a_0, a_1)$. Since $\tilde{P}_b(y)$ is monic and of degree $r > 0$, it follows that $\tilde{P}_b(x_0, y) = 0$ has a positive real root in $y$ for all $x_0 \in (-a_0, a_1)$.

We now consider the case that $x_0 = -a_0$ or $x_0 = a_1$. As noted earlier, for each $j \in \{1, 2, \ldots, r\}$, the polynomial $p_j(b, x)$ has its roots in the interval $[-\hat{a}_0, \hat{a}_1]$ and $p_j(b, \pm 11) > 0$. Since each of $-a_0, a_1$ and $\pm 11$ is not in $[-\hat{a}_0, \hat{a}_1]$ while $x_0 = -a_0$ or $x_0 = a_1$, it follows that $p_j(b, x_0) > 0$ for each such $j$. From Descartes’ rule of signs, we deduce that $\tilde{P}_b(x_0, y)$ has no positive real roots. Thus, $\tilde{P}_b(x_0, y)$ has 0 as its largest real root.

For a given $x \in I_b$, we now define $\rho_b(x)$ as the largest non-negative real root of $P_b(x, y)$. The above arguments show that $\rho_b(x)$ is well-defined.

For each $x \in J_b$, define

$$\psi_b(x) = \max \left\{ y \in \mathbb{R} : \tilde{P}_b(y) = 0 \right\}.$$

Since $\tilde{P}_b(y)$ has real roots for any given $x \in J_b$, then $\psi_b(x)$ is well-defined. Moreover, we have now seen that $\psi_b(x) > 0$ for all $x \in (-a_0, a_1)$, and $\psi_b(-a_0) = \psi_b(a_1) = 0$. Parts (ii) and (iii) now follow by observing that $\rho_b(x) = \sqrt{\psi_b(x - b)}$ for each $x \in I_b$.

Next, we turn to the arguments for parts (iv) and (v). The arguments for these parts are similar to the proofs of part (d) and (e) of Lemma 2 in [7].
To prove $\rho_b(x)$ is a continuously differentiable function on $(b - a_0, b + a_1)$, it is sufficient to show that, given any $x_0 \in (-a_0, a_1)$, there exists an open interval $J' \subseteq (-a_0, a_1)$ containing $x_0$ such that $\psi_b(x)$ is a continuously differentiable function on $J'$. To prove that $\rho_b(x)$ is a continuous function on $[b - a_0, b + a_1]$, we will also want to show that

$$\lim_{x \to -a_0^+} \psi_b(x) = 0 \quad \text{and} \quad \lim_{x \to a_1^-} \psi_b(x) = 0.$$  

Fix $x_0 \in (-a_0, a_1)$, and let $y_0 = \psi_b(x_0)$. We make use of Lemma 3.2 with $W(x, y) = \widehat{P}_b(x, y)$. Since then $W(x, y)$ is a polynomial, both $W_x$ and $W_y$ are continuous on all of $\mathbb{R}^2$. The definition of $y_0$ implies $W(x_0, y_0) = 0$.

For Lemma 3.2, we want to also show that $W_y(x_0, y_0) \neq 0$. In the case that $b \neq 2$, we calculate the discriminant $\Delta_b(x)$ of $\widehat{P}_b(y)$. A Sturm sequence computation shows that $\Delta_b(x) \neq 0$ for all $x \in \mathbb{R}$. To clarify, the computation of the Sturm sequence was shortened by first factoring the discriminant and then showing $\Delta_b(x) \neq 0$ for all $x \in \mathbb{R}$ by establishing that each factor of $\Delta_b(x)$ is non-zero for all $x \in \mathbb{R}$ using a separate Sturm sequence for each factor. Therefore, in the case that $b \neq 2$, we have that $\widehat{P}_b(x_0, y_0)$ has no repeated roots, so $W_y(x_0, y_0) \neq 0$.

In the case that $b = 2$, a Sturm sequence computation shows that $\Delta_2(x)$ is non-zero on $J_2$ when $x \neq -1/2$. Thus, we have that $\widehat{P}_2(x, y)$ has a repeated root for $x \in J_2$ only when $x = -1/2$. By factoring $\widehat{P}_2(-1/2, y)$, one sees that the only repeated root of $\widehat{P}_2(-1/2, y)$ is $y = -1/4$. Therefore, in our case where $y_0 \geq 0$, $W_y(x_0, y_0) \neq 0$.

Now define $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : -a_0 < x < a_1 \text{ and } y > 0\}$. By Lemma 3.2, there exist an open interval $J'' \subseteq (-a_0, a_1)$ containing $x_0$ and a continuously differentiable function $\phi(x)$ defined on $J''$ such that both $\phi(x_0) = y_0$ and $\hat{P}_b(x, \phi(x)) = 0$ for all $x \in J''$. By the definition of $\psi_b(x)$, we know that $\phi(x) \leq \psi_b(x)$ for all $x \in J''$. We will show that there exists an open interval $J' \subseteq J''$ containing $x_0$ such that $\psi_b(x) = \phi(x)$ for all $x \in J'$.

By way of contradiction, assume that no such interval $J'$ exists. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ satisfying $\lim_{n \to \infty} x_n = x_0$ and having the property that, for all $n \geq 1$, $\psi_b(x_n) > \phi(x_n)$. Since $x_0 \in J''$, we suppose further as we may that each $x_n \in J''$. Define $y_n = \psi_b(x_n)$. In particular, $\hat{P}_b(x_n, y_n) = 0$.

We justify that $\{y_n\}_{n=1}^{\infty}$ is a bounded sequence. In fact, we show that there is an absolute constant $M$ such that for $x' \in J_b$ and $z \in \mathbb{C}$ satisfying $\hat{P}_b(x', z) = 0$, we have $|z| \leq M$. Since each $p_j(b, x)$ is continuous on $J_b$ and $J_b$ is compact, there exists an absolute constant $A \geq 0$ such that $|p_j(b, x)| \leq
A for all \( j \in \{0, \ldots, r\} \) and \( x \in J_b \). Recall \( p_r(b, x) \equiv 1 \). Since \( x' \in J_b \) and \( \overrightarrow{P}_b(x', z) = 0 \), we deduce
\[
0 = \left| \sum_{j=0}^{r} p_j(b, x') z^j \right| \geq |z|^r - \sum_{j=0}^{r-1} |p_j(b, x')| |z|^j \geq |z|^r - A \sum_{j=0}^{r-1} |z|^j.
\]
Thus, \( |z| \) is less than or equal to the positive real root \( M \) of the polynomial
\[
x^r - Ax^{r-1} - Ax^{r-2} - \cdots - Ax - A.
\]
We deduce that \( \{y_n\}_{n=1}^{\infty} \) is a sequence with \( |y_n| \leq M \) for all \( n \).

It follows now that the sequence \( \{y_n\}_{n=1}^{\infty} \) has a convergent subsequence \( \{y_{n_j}\}_{j=1}^{\infty} \). Let \( L = \lim_{j \to \infty} y_{n_j} \). The continuity of \( \overrightarrow{P}_b(x, y) \) implies
\[
\overrightarrow{P}_b(x_0, L) = \lim_{j \to \infty} \overrightarrow{P}_b(x_{n_j}, y_{n_j}) = 0.
\]
Since
\[
y_0 = \psi_b(x_0) = \max\{y \in \mathbb{R} : \overrightarrow{P}_b(x_0, y) = 0\},
\]
we deduce that \( L \leq y_0 \). Since \( \phi(x) \) is continuous on \( J'' \) and \( \phi(x_{n_j}) \leq \psi_b(x_{n_j}) = y_{n_j} \) for all \( j \geq 1 \), we also have that
\[
L = \lim_{j \to \infty} y_{n_j} = \lim_{j \to \infty} \psi_b(x_{n_j}) \geq \lim_{j \to \infty} \phi(x_{n_j}) = \phi\left( \lim_{j \to \infty} x_{n_j} \right) = \phi(x_0) = y_0.
\]
Thus, \( L = y_0 \). In particular,
\[
(3.7) \quad \lim_{j \to \infty} \psi_b(x_{n_j}) = y_0 = \lim_{j \to \infty} \phi(x_{n_j}).
\]
We show that this implies a contradiction.

Consider
\[
|W(x_{n_j}, \psi_b(x_{n_j})) - W(x_{n_j}, \phi(x_{n_j}))| = 0.
\]
By the Mean Value Theorem, we have that
\[
(3.8) \quad |\psi_b(x_{n_j}) - \phi(x_{n_j})| |W_y(x_{n_j}, \xi_j)| = 0
\]
for some \( \xi_j \in [\phi(x_{n_j}), \psi_b(x_{n_j})] \). Since \( \psi_b(x_{n_j}) > \phi(x_{n_j}) \), we have from (3.8) that
\[
W_y(x_{n_j}, \xi_j) = 0.
\]
Taking the limit as \( j \to \infty \), we have by (3.7) that \( \lim_{j \to \infty} \xi_j = y_0 \) so that \( W_y(x_0, y_0) = 0 \). But this contradicts the fact that \( W_y(x_0, y_0) \neq 0 \). Therefore, there exists an open interval \( J' \subseteq J'' \) containing \( x_0 \) such that \( \psi_b(x) = \phi(x) \) for all \( x \in J' \).

To finish the proof of part (iv), we need only to show that \( \psi_b(x) \) is continuous at the endpoints of \( J_b \). Let \( \{x_n\}_{n=1}^{\infty} \subset J_b \) be a sequence that
converges to one of the endpoints of $J_b$, say $a'$. Take $y_n = \psi_b(x_n)$. With $M$ as before, we have that $|y_n| \leq M$. To show that
\[
\lim_{n \to \infty} \psi_b(x_n) = 0 = \psi_b(a'),
\]
it suffices to prove that every convergent subsequence of $y_n$ converges to 0.

Suppose that $\{y_n\}$ is such that $\lim_{j \to \infty} y_{n_j} = L$ for some $L \in \mathbb{R}$. Since we know that $y_{n_j} = \psi_b(x_{n_j}) \geq 0$, we deduce $0 \leq L \leq M$. Now,
\[
\hat{P}_b(a', L) = \lim_{j \to \infty} \hat{P}_b(x_{n_j}, y_{n_j}) = \lim_{j \to \infty} \hat{P}_b(x_{n_j}, \psi_b(x_{n_j})) = 0.
\]
Therefore, $L \leq \psi_b(a') = 0$. Hence, $L = 0$, completing the proof of part (iv).

To establish part (v), we first observe that the definition of $\rho_b(x)$ implies that if $x \in I_b$ and $y \in \mathbb{R}$ are such that $P_b(x, y) = 0$, then $|y| \leq \rho_b(x)$. Part (i) also implies if $P_b(x, y) = 0$ for some real numbers $x$ and $y$, then $x \in I_b$.

Now, consider real numbers $x_0$ and $y_0$ for which $P_b(x_0, y_0) < 0$. Note that (3.6) implies $x_0 \in I_b$ and $P_b(0, 0) > 0$. Since $P_b(x, y)$ is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, we deduce that along any path from $(0, 0)$ to $(x_0, y_0)$ in $\mathbb{R}^2$, there must be a point $(x, y)$ satisfying $P_b(x, y) = 0$. We use again that for any $x \in J_b$, the number $M$ is a bound on the absolute value of the roots of $\hat{P}_b(y)$. We deduce from (3.5) that $\rho_b(x) \leq \sqrt{M}$ for all $x \in I_b$. If $x_0 \in I_b$ and $y_0 > \rho_b(x_0)$, then one can consider the path consisting of line segments from $(0, 0)$ to $\left(0, 1 + \sqrt{M}\right)$, from $\left(0, 1 + \sqrt{M}\right)$ to $\left(x_0, 1 + \sqrt{M}\right)$ and from $\left(x_0, 1 + \sqrt{M}\right)$ to $\left(x_0, y_0\right)$ to obtain a contradiction. If $x_0 \in I_b$ and $y_0 < -\rho_b(x_0)$, one can consider a similar path but from $(0, 0)$ to $\left(0, -1 - \sqrt{M}\right)$ to $\left(x_0, -1 - \sqrt{M}\right)$ to $\left(x_0, y_0\right)$ to obtain a contradiction. Therefore, we must have $x_0 \in I_b$ and $|y_0| \leq \rho_b(x_0)$. This establishes part (v), completing the proof.

Now that we have proven Lemma 3.1, we will use it in the next sections to prove irreducibility criteria based on the degree of $f(x)$ and on the size of the coefficients of $f(x)$.

4. Irreducibility Criteria Based on Degree

Fix an integer $b \in [2, 20]$. Let $f(x) \in \mathbb{Z}[x]$ have non-negative coefficients, with $f(b)$ prime. Theorem 2.2 in Section 2 led us to deduce the irreducibility of $f(x)$ given bounds $D(b)$ on the degree of $f(x)$. As noted there, those bounds were not necessarily sharp. In this section, we use the region $\mathcal{R}_b$ to establish sharp bounds corresponding to Theorem 2.2.
Take for example $b = 6$. Theorem 2.2 and Table 3 give us that if $f(6)$ is prime and the degree of $f(x)$ is $\leq 18$, then $f(x)$ is irreducible. We now show that if $f(6)$ is prime and the degree of $f(x)$ is $\leq 19$, then $f(x)$ is irreducible. Furthermore, we give an example to show that this bound is sharp.

Our next lemma follows from the proof of Theorem 2.2 given in Section 1.

**Lemma 4.1.** Let $n$ be a positive integer. A complex number $\alpha = re^{i\theta}$, such that $0 < \theta < \pi/n$, cannot be a root of a non-zero polynomial with non-negative integer coefficients and degree $\leq n$.

Now, we can establish the following improvement on Theorem 2.2.

**Theorem 4.2.** Fix an integer $b \in [2, 20]$, and let $D = D(b)$, $D_1 = D_1(b)$, and $D_2 = D_2(b)$ be as in Table 7. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_j \geq 0$ for each $j$ and with $f(b)$ prime. If the degree of $f(x)$ is $\leq D$, then $f(x)$ is irreducible. Additionally, if the degree of $f(x)$ is $\leq D_1$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_4(x-b)$ and not divisible by $\Phi_3(x-b)$. Furthermore, if the degree of $f(x)$ is $\leq D_2$ and $f(x)$ is reducible, then $f(x)$ is divisible by either $\Phi_4(x-b)$ or $\Phi_3(x-b)$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$D(b)$</th>
<th>$D_1(b)$</th>
<th>$D_2(b)$</th>
<th>$\theta(b)$</th>
<th>$m(b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 2$</td>
<td>6</td>
<td>-</td>
<td>7</td>
<td>$\pi/7$</td>
<td>13/27</td>
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<td>$b = 3$</td>
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<td>-</td>
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<td>17/80</td>
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<td>70/397</td>
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<td>$\pi/44$</td>
<td>1/14</td>
</tr>
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<td>$b = 13$</td>
<td>40</td>
<td>45</td>
<td>47</td>
<td>$\pi/47$</td>
<td>1/15</td>
</tr>
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<td>44</td>
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<td>51</td>
<td>$\pi/51$</td>
<td>4/65</td>
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<td>$b = 15$</td>
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<tr>
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<td>62</td>
<td>70</td>
<td>72</td>
<td>$\pi/72$</td>
<td>1/23</td>
</tr>
</tbody>
</table>

**Table 7.** $D(b)$, $D_1(b)$, $D_2(b)$, $\theta(B)$, and $m(b)$ for $b \in [2, 20]$. 
We note that for \( b \in \{2, 3, 4\} \), there is no value for \( D_1 \) due to the equality
\[
\left\lfloor \frac{\pi}{\arg (b + \zeta_4)} \right\rfloor = \left\lfloor \frac{\pi}{\arg (b + \zeta_3)} \right\rfloor \quad \text{for} \quad b \in \{2, 3, 4\}.
\]

By way of examples, we will demonstrate later that the values of \( D(b) \) and \( D_1(b) \) given in Table 7 are sharp. We do not know that this is the case for the values of \( D_2(b) \). It is also worth noting that \( D_2(10) \) above is an improvement over the value 36 established in [7].

Proof of Theorem 4.2. Following the remarks before the proof of Lemma 3.1, we set
\[
(e_2, e_3, e_4, e_6, d) = \begin{cases} 
(0, 2, 1, 0, 1) \quad &\text{for} \quad b = 2, \\
(0, 2, 3, 0, 8) \quad &\text{for} \quad b = 3, \\
(0, 2, 4, 0, 8) \quad &\text{for} \quad b = 4 \text{ or } 5 \\
(0, 2, 5, 0, 12) \quad &\text{for} \quad b = 6 \text{ or } 7 \\
(0, 1, 8, 0, 14) \quad &\text{for} \quad 8 \leq b \leq 14 \\
(0, 1, 10, 0, 24) \quad &\text{for} \quad 15 \leq b \leq 20.
\end{cases}
\]

We define \( F_b(z) \) as in (3.1), \( P_b(x, y) \) as in (3.3), and \( R_b \) as in (3.2). In addition to \( D = D(b) \), \( D_1 = D_1(b) \) and \( D_2 = D_2(b) \), we set \( \vartheta = \vartheta(b) \) and \( m = m(b) \) as in Table 7. We note that \( m \) is a rational number.

We consider the line \( y = \tan (\vartheta) x \) or equivalently the points \( x + i \tan (\vartheta) x \) in the complex plane. A simple computation gives us that \( \tan (\vartheta) > m \). So the line \( y = mx \) lies strictly below the line \( y = \tan (\vartheta) x \) for \( x > 0 \). Applying Lemma 3.1, we know that \( \rho_b(b - a_0) = 0 \) and that \( \rho_b(x) \) is continuous. We use a Sturm sequence to verify that \( P_b(x, mx) \) has no real roots. Since the coefficients of \( P_b(x, mx) \) are rational, this computation involves only exact arithmetic. Using Lemma 3.1 part (ii), we can deduce that \( R_b \) does not intersect the line \( y = mx \). Therefore, the entire region \( R_b \) lies below the line \( y = mx \).

We recall the set-up from Section 3. We suppose \( f(x) \) is reducible and write \( f(x) = g(x)h(x) \), where both \( g(x) \) and \( h(x) \) are in \( \mathbb{Z}[x] \), \( g(x) \neq \pm 1, h(x) \neq \pm 1 \), and both \( g(x) \) and \( h(x) \) have positive leading coefficients. Furthermore, without loss of generality, we suppose that \( g(b) = \pm 1 \). In Section 3, we showed that either \( g(x) \) has a root in common with at least one of \( \Phi_3(x - b), \Phi_4(x - b) \) and \( \Phi_6(x - b) \), or \( g(x) \) has a root \( \beta \in R_b \). Since \( f(x) \) has non-negative coefficients and the real numbers in \( R_b \) are positive, we know that \( \beta \not\in \mathbb{R} \).
With our choices above, \( b + \zeta_6 \) lies below the line \( y = mx \) for each \( b \in [2, 20] \). This is illustrated in Figure 6 for \( b = 5 \), where the straight line passes through the origin and its slope is \( \frac{70}{397} \).

![Figure 6. \( y = 70x/397 \) above \( R_5 \) and \( 5 + \zeta_6 \)]

We conclude that either \( g(x) \) has a root in common with at least one of \( \Phi_3(x - b) \) and \( \Phi_4(x - b) \), or \( g(x) \) has a root \( \beta = \sigma + it \) such that \( 0 < t < m\sigma < \tan(\vartheta) \sigma \). Note that the latter implies that if \( \beta = re^{i\vartheta} \), then \( \vartheta' < \vartheta \).

With an eye toward applying Lemma 4.1, we deduce from Table 7 that if \( \vartheta' < \vartheta = \pi/D \) for \( b \in \{2, 3, 4\} \) and \( \vartheta' < \vartheta = \pi/D_2 < \pi/D_1 < \pi/D \) for \( b \in [5, 20] \).

For \( b \geq 3 \), a computation gives \( \arg(b + \zeta_3) < \pi/D \) and \( \arg(b + \zeta_4) < \pi/D \). Thus, by Lemma 4.1, we have that \( f(x) \) is irreducible if \( \deg f \leq D \).

In the case of \( b = 2 \), we have that \( \arg(2 + \zeta_4) < \pi/D \) but arg \( (2 + \zeta_3) = \pi/D \). We show that in this case, if \( \deg f(x) = D = 6 \) and \( f(x) \) is divisible by \( \Phi_3(x - 2) \), then \( f(2) \) is necessarily composite, contradicting our original assumption.

Since we want \( \Phi_3(x - 2) = x^2 - 3x + 3 \) to be a factor of \( f(x) \), and \( \deg f(x) = 6 \), we know that the other factor of \( f(x) \) is \( u_1 x^4 + u_2 x^3 + u_3 x^2 + u_4 x + u_5 \), where \( u_1, u_2, u_3, u_4, u_5 \in \mathbb{Z} \) and \( u_1 \geq 1 \). This gives us that

\[
f(x) = (x^2 - 3x + 3) (u_1 x^4 + u_2 x^3 + u_3 x^2 + u_4 x + u_5) \\
= u_1 x^6 + (u_2 - 3 u_1) x^5 + (3 u_1 - 3 u_2 + u_3) x^4 + (3 u_2 - 3 u_3 + u_4) x^3 \\
+ (3 u_3 - 3 u_4 + u_5) x^2 + (3 u_4 - 3 u_5) x + 3 u_5.
\]

Observe that \( 2 + \zeta_3 \) is a root of \( f(x) \) and each coefficient of \( f(x) \) is non-negative. Also, the imaginary part of \( (2 + \zeta_3)^j \) is \( > 0 \) for \( j \in \{1, 2, 3, 4, 5\} \), and \( (2 + \zeta_3)^6 = -27 \). If one of the coefficients of \( x, x^2, x^3, x^4 \) or \( x^5 \) in \( f(x) \) is \( > 0 \), then \( \text{Im}(f(2 + \zeta_3)) > 0 \), contradicting the fact that \( 2 + \zeta_3 \) is a root of \( f(x) \). Thus, we have that \( u_2 - 3 u_1 = 0, 3 u_1 - 3 u_2 + u_3 = 0, 3 u_2 - 3 u_3 + u_4 = 0, 3 u_3 - 3 u_4 + u_5 = 0 \) and \( 3 u_4 - 3 u_5 = 0 \). Solving for \( u_2, u_3, u_4 \) and \( u_5 \), we obtain \( u_2 = 3 u_1, u_3 = 6 u_1, u_4 = 9 u_1 \) and \( u_5 = 9 u_1 \). This gives us that
\[ f(x) = u_1 x^6 + 27u_1. \] Hence, \( f(2) = 91u_1 = 7 \times 13 \times u_1, \) so \( f(2) \) is composite. Thus, the case \( b = 2 \) also leads to the statement involving the bound \( D \) in Theorem 4.2.

We now turn to establishing the statements concerning \( D_1 \) and \( D_2 \).

For \( b \geq 5 \), we have that \( \arg (b + \zeta_3) < \pi/D_1 \), \( \arg (b + \zeta_4) > \pi/D_1 \), and \( D_1 > D \). Thus, by Lemma 4.1, we have that if \( f(x) \) is reducible and deg \( f(x) \leq D_1 \), then \( f(x) \) is divisible by \( \Phi_4(x - b) \). For \( 2 \leq b \leq 20 \), we have \( \arg (b + \zeta_3) > \pi/D_2 \), \( \arg (b + \zeta_4) > \pi/D_2 \), and \( D_2 > D \). Thus, by Lemma 4.1, we have that if \( f(x) \) is reducible and the degree of \( f(x) \) is \( \leq D_2 \), then \( f(x) \) is divisible by \( \Phi_3(x - b) \) or \( \Phi_4(x - b) \). Note that what is significant, in this part of the argument, is that \( \tan(\pi/D_2) \geq m \) and \( y = mx \) lies above the region \( R_b \).

This completes the proof. \( \square \)

Examples given later in Table 19 and Table 20 will show that the bounds \( D(b) \) and \( D_1(b) \) are sharp. For example, take \( b = 6 \), where we see that \( D(6) \) has increased from 18 in Theorem 2.2 to 19 in Theorem 4.2. The polynomial

\[ f(x) = x^{20} + 2x^3 + 13519269991320x^2 + 610418402115746x + 610418402115527 \]

is of degree 20, \( f(6) = 8415780974560931 \) is prime, each coefficient of \( f(x) \) is at most 610418402115745, and \( f(x) \) is divisible by \( \Phi_4(x - 6) = x^2 - 12x + 37 \). Although not our ultimate goal, we will prove later in Section 8 that this polynomial is also optimal in terms of the size of its coefficients. We will show that if \( f(x) \in \mathbb{Z}[x] \) is a polynomial of degree 20 with non-negative integer coefficients which are \( \leq 610418402115745 \) and \( f(6) \) is prime, then \( f(x) \) is irreducible. More generally, we will establish the following result.

**Theorem 4.3.** Fix an integer \( b \in [2, 20] \), let \( D = D(b) \) and \( D_1 = D_1(b) \) (for \( b \geq 5 \)) be as in Table 7, let \( N_1 = N_1(b) \) be as in Table 8, and let \( N_2 = N_2(b) \) (for \( b \geq 5 \)) be as given in Table 9. Let \( f(x) = \sum_{j=0}^{D} a_j x^j \in \mathbb{Z}[x] \) be such that \( a_j \geq 0 \) for each \( j \) and \( f(b) \) is prime. If deg \( f(x) = D + 1 \) and each \( a_j \leq N_1 \), then \( f(x) \) is irreducible. In the case that \( 5 \leq b \leq 20 \), if deg \( f(x) = D_1 + 1 \) and each \( a_j \leq N_2 \), then \( f(x) \) is either irreducible or divisible by \( \Phi_3(x - b) \) if \( b = 5 \) or divisible by \( \Phi_4(x - b) \) if \( b \in [6, 20] \).

As indicated before, the bounds \( N_1(b) \) and \( N_2(b) \) given in Table 8 and Table 9 will all be shown to be sharp and will involve coming up with explicit examples. These details appear in Section 8.
<table>
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<th>$N_1(b)$</th>
</tr>
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<tbody>
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Table 8. $N_1(b)$ for $2 \leq b \leq 20$
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Table 9. $N_2(b)$ for $5 \leq b \leq 20$
5. A First Bound on the Coefficients

Throughout this section, $\mathcal{R}_b$ is as defined in (3.2), with $F_b(z)$ given by (3.1) and $P_b(x, y)$ given by (3.3). The numbers $e_2(b), e_3(b), e_4(b), e_6(b)$ and $d(b)$ are as given in Table 4.

We summarize the previous sections and set the goal for this section. We have fixed an integer $b \in [2, 20]$, and taken a polynomial $f(x)$ with each coefficient of $f(x)$ non-negative and $f(b)$ prime. We considered $f(x) = g(x)h(x)$, with $g(x) \not\equiv \pm 1$, $h(x) \not\equiv \pm 1$, and both $g(x)$ and $h(x)$ having positive leading coefficients. Using that $f(b)$ is prime, we reduced our considerations to $g(b) = \pm 1$. We then showed that either $g(x)$, and thus $f(x)$, has a factor of at least one of $\Phi_3(x - b)$, $\Phi_4(x - b)$ and $\Phi_6(x - b)$, or $g(x)$ has a root $\beta \in \mathbb{R}_b$.

Now we consider the latter case, that $g(x)$, and thus $f(x)$, has a root $\beta \in \mathbb{R}_b$ and obtain a lower bound on the coefficients of $f(x)$ in this case. We will rely heavily on the following lemma.

**Lemma 5.1.** Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$, where $a_j \geq 0$ for $j \in \{0, 1, \ldots, n\}$. Suppose $\alpha = r e^{i \theta}$ is a root of $f(x)$ with $0 < \theta < \pi/2$ and $r > 1$. Let

$$B = \max_{\pi/(2\theta) < k < \pi/\theta} \left\{ \frac{r^k (r - 1)}{1 + \cot(\pi - k\theta)} \right\},$$

where the maximum is over $k \in \mathbb{Z}$. Then there is some $j \in \{0, 1, \ldots, n - 1\}$ such that $a_j > Ba_n$.

The proof of Lemma 5.1 is similar to the proof of Theorem 5 in [6] and is established in the above form in [7] (cf., [1] and [2]).

We use Lemma 5.1 to prove the following Corollary.

**Corollary 5.2.** Fix an integer $b$ with $b \geq 2$. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be such that $a_j \geq 0$ for each $j$ and $f(b)$ is prime. If

$$0 \leq a_j \leq B_b a_n \quad \text{for} \quad 0 \leq j \leq n - 1 \quad \text{with} \quad B_b \text{ as in Table 10},$$

then either $f(x)$ is irreducible or $f(x)$ is divisible by at least one of $\Phi_3(x - b)$, $\Phi_4(x - b)$ and $\Phi_6(x - b)$.

Before proceeding to the argument for Corollary 5.2, we note that the value for $B_{10}$ given in Table 10 is an improvement over the analogous result given in [7]. This is due to our choice of $e_2(10), e_3(10), e_4(10), e_6(10)$ and $d(10)$ in Table 4, which differs from that used in [7]. On the other hand, the methods we use to obtain Corollary 5.2 are analogous to what was used to obtain a similar bound for $b = 10$ in [7].
Proof of Corollary 5.2. For a fixed integer $b \in [2, 20]$, let $\theta$ and $\theta'$ be real numbers such that $0 \leq \theta < \theta' \leq \tan^{-1}(R_b)$, where $R_b$ is given in Table 11. We are interested in the set of points $R_b(\theta, \theta')$ that are in $R_b$ between the line passing through the origin making an angle $\theta$ with the positive $x$-axis and the line passing through the origin making an angle $\theta'$ with the positive $x$-axis. Explicitly, we define

$$R_b(\theta, \theta') = \{(x, y) \in R_b : \tan \theta \leq y/x < \tan \theta'\}.$$  

We are still considering the case that $g(x)$ has a root $\beta \in R_b$. We write $\beta = x_0 + iy_0$ for some $(x_0, y_0) \in R_b$, where we may take $y_0 > 0$.

Along the lines of the proof of Theorem 4.2, we use a Sturm sequence to show that the line $y = R_b x$ does not intersect the region $R_b$, where the value of $R_b$ is given in Table 11. In other words, we take the rational equivalent of the decimal expression in Table 11 and show that the region $R_b$ lies completely under the line $y = R_b x$ by verifying with a Sturm sequence that the polynomial $P_b(x, R_b x) \in \mathbb{Q}[x]$ has no real roots.

<table>
<thead>
<tr>
<th>$b$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_b$</td>
<td>1.6</td>
<td>0.5</td>
<td>0.26</td>
<td>0.18</td>
<td>0.15</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.124</td>
<td>0.108115</td>
<td>0.096</td>
<td>0.08622</td>
<td>0.0783</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.072</td>
<td>0.0664</td>
<td>0.0617</td>
<td>0.0577</td>
<td>0.054053</td>
</tr>
<tr>
<td>$R_b$</td>
<td>0.05091</td>
<td>0.0481</td>
<td>0.0456</td>
<td>0.043327</td>
<td></td>
</tr>
</tbody>
</table>

Table 11. Values of $R_b$
To utilize Lemma 5.1, we specify a set \( \Theta_b = \{ \theta_0, \theta_1, \ldots, \theta_{m-1}, \theta_m \} \) where
\[
0 = \theta_0 < \theta_1 < \ldots < \theta_{m-1} < \theta_m < \pi/2,
\]
and where \( \tan(\theta_l) = r_l \in \mathbb{Q} \) for \( 0 \leq l \leq m \), \( \tan(\theta_1) = 1/1000 \) and \( \tan(\theta_m) = R_b \). Thus, we have that
\[
(x_0, y_0) \in \bigcup_{l=0}^{m-1} R_b(\theta_l, \theta_{l+1}).
\]

Next, for each \( l \in \{0, 1, \ldots, m - 1\} \), we use Lemma 5.1 to find a bound \( B'_b(\theta_l, \theta_{l+1}) \) so that for all \( (x_0, y_0) \in R_b(\theta_l, \theta_{l+1}) \), there is a \( j \in \{0, 1, \ldots, n - 1\} \) for which \( a_j > B'_b(\theta_l, \theta_{l+1})a_n \). We can then deduce that some coefficient of \( f(x) \) must exceed
\[
\min_{0 \leq l \leq m-1} \{ B'_b(\theta_l, \theta_{l+1}) \} \cdot a_n. \tag{5.1}
\]

We judiciously choose each \( \theta_l \in \Theta_b \) so that each \( B'_b(\theta_l, \theta_{l+1}) > B_b \), where \( B_b \) is listed in Table 10. Corollary 5.2 will then follow.

We begin by considering the first sector \( R_b(\theta_0, \theta_1) \), where we have already stated that \( \theta_0 = 0 \) and \( \theta_1 = \tan^{-1}(1/1000) \), independent of the value of \( b \in [2, 20] \). Take
\[
k = k(\theta) = \left\lfloor \frac{25\pi}{26\theta} \right\rfloor \text{ where } 0 < \theta \leq \tan^{-1}\left( \frac{1}{1000} \right).
\]

We note that
\[
k \in \left( \frac{\pi}{2\theta}, \frac{\pi}{\theta} \right)
\]

since
\[
k\theta \leq \frac{25\pi}{26} < \pi
\]

and
\[
k\theta > \left( \frac{25\pi}{26\theta} - 1 \right) \theta = \frac{25\pi}{26} - \theta \geq \frac{25\pi}{26} - \tan^{-1}\left( \frac{1}{1000} \right) > \frac{\pi}{2}
\]

We will use later that
\[
\frac{\pi}{2} > \pi - k\theta \geq \pi - \frac{25\pi}{26} = \frac{\pi}{26},
\]

which gives us that \( \cot(\pi - k\theta) \leq \cot(\pi/26) \).

From our definition of \( k \) and the range of \( \theta \) above, we have that
\[
k = \left\lfloor \frac{25\pi}{26\theta} \right\rfloor \geq \left\lfloor \frac{25\pi/26}{\tan^{-1}(1/1000)} \right\rfloor = 3020.
\]

We recall that for each \( z \in R_b \), regardless of the \( b \) we are using, we have that the \( \Re(z) \geq 1.447 \), as implied by Table 6. Thus, for each \( z = re^{i\theta} \in R_b \), we have that \( r = |z| \geq 1.447 \). For each such \( z \), we have that
\[
\frac{r^k(r - 1)}{1 + \cot(\pi - k\theta)} \geq \frac{1.447^{3020}(1.447 - 1)}{1 + \cot(\pi/26)} > 1.99 \times 10^{483}.
\]
From Lemma 5.1, with \( \theta_0 = 0 \) and \( \theta_1 = \tan^{-1}(1/1000) \), we see that we may take

\[
B'_b(\theta_0, \theta_1) = B'_b\left(0, \tan^{-1}\left(\frac{1}{1000}\right)\right) = 1.99 \times 10^{483}.
\]

Observe that \( B'_b(\theta_0, \theta_1) = 1.99 \times 10^{483} \) is larger than \( B_b \) for each \( b \in [2, 20] \).

There is quite a bit of freedom in choosing the remaining values of \( \theta_l \) for each \( b \). We want some idea of where the line \( y = \tan(\theta_l) x \) intersects \( \mathcal{R}_b \). Since the boundary of \( \mathcal{R}_b \) consists of the points \((x, y)\) such that \( P_b(x, y) = 0 \), we want an estimate of the real numbers \( x \) for which \( P(x, \tan(\theta_l) x) = 0 \).

However, we want to avoid computations that approximate the real roots of a polynomial based on coefficients that are themselves just approximations of the actual real coefficients. To this end, we recall \( r_l = \tan(\theta_l) \), where \( r_l \) is a rational number. We then find a close rational lower bound approximation \( x'_l \) to the minimum real root of \( P_b(x, r_l x) = 0 \). Since \( P_b(x, r_l x) \in \mathbb{Q}[x] \) and \( x'_l \in \mathbb{Q} \), we can then use a Sturm sequence to verify, with exact arithmetic, that \( P_b(x, r_l x) \) has no roots in the interval \([0, x'_l]\). Thus, \( x'_l \) provides us with a lower bound on the \( x \)-coordinate of the intersection of \( y = \tan(\theta_l) x \) with \( \mathcal{R}_b \).

Observe that by construction, \( r_1 = 1/1000 \).

The values of \( r_l = \tan(\theta_l) \) we used for each \( b \in [2, 20] \) can be found in [4]. As the exact values are not so significant, we do not duplicate them all here but rely instead on tabulating the choices we used for \( b = 2 \) and \( b = 10 \) as examples. For \( b = 2 \), the \( r_l \) are given in Table 12; for \( b = 10 \), the \( r_l \) are given in Table 13.

We explain the notation in Table 13 for the values of \( \theta_0, \theta_1, \ldots, \theta_m \). The value \( r_a \) corresponds to the first value of \( \tan(\theta_l) \) being considered in that row, and the value \( r_b \) corresponds to the last value of \( \tan(\theta_{l+1}) \). We used the rational equivalents of the decimals given for \( r_a \) and \( r_b \) in our computations to ensure exact arithmetic when computing \( x'_l \) as described earlier. If \( d \) is the number of divisions indicated in the third column of the same row, then the corresponding intervals \((\theta_l, \theta_{l+1})\) for that row are given by

\[
\theta_l = \tan^{-1}\left(r_a + \frac{(r_b - r_a)j}{d}\right),
\]

\[
\theta_{l+1} = \tan^{-1}\left(r_a + \frac{(r_b - r_a)(j + 1)}{d}\right), \quad \text{for} \ 0 \leq j \leq d - 1,
\]

where \( l \) as indicated depends on \( j \). The fourth column indicates the minimum value of \( B'_{10}(\theta_l, \theta_{l+1}) \) for \((\theta_l, \theta_{l+1})\) considered in that row and, therefore, serves as a value of \( B'_{10}(\theta_a, \theta_b) \). We explain momentarily how the bounds \( B'_{b}(\theta_l; \theta_{l+1}) \) were obtained. The number \( m \) of intervals \((\theta_l, \theta_{l+1})\) for \( b = 10 \)
<table>
<thead>
<tr>
<th>$l$</th>
<th>$r_l = \tan (\theta_l)$</th>
<th>$B'<em>2 (\theta_l, \theta</em>{l+1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 = 0</td>
<td>$1.99 \times 10^{483}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{1000}$ = 0.001</td>
<td>$1.67316 \times 10^{333}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{3}{2000}$ = 0.0015</td>
<td>$1.88152 \times 10^{249}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{500}$ = 0.002</td>
<td>$1.78851 \times 10^{165}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{3}{1000}$ = 0.003</td>
<td>$2.25395 \times 10^{123}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{1}{250}$ = 0.004</td>
<td>$1.59285 \times 10^{98}$</td>
</tr>
<tr>
<td>6</td>
<td>$\frac{1}{200}$ = 0.005</td>
<td>$3.13071 \times 10^{81}$</td>
</tr>
<tr>
<td>7</td>
<td>$\frac{3}{500}$ = 0.006</td>
<td>$3.66576 \times 10^{69}$</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{7}{1000}$ = 0.007</td>
<td>$3.99316 \times 10^{60}$</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{1}{125}$ = 0.008</td>
<td>$3.51475 \times 10^{53}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{9}{1000}$ = 0.009</td>
<td>$1.01194 \times 10^{48}$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{1}{100}$ = 0.01</td>
<td>$2.52294 \times 10^{31}$</td>
</tr>
<tr>
<td>12</td>
<td>$\frac{3}{200}$ = 0.015</td>
<td>$1.13455 \times 10^{23}$</td>
</tr>
<tr>
<td>13</td>
<td>$\frac{1}{50}$ = 0.02</td>
<td>$8.071030 \times 10^{14}$</td>
</tr>
<tr>
<td>14</td>
<td>$\frac{3}{100}$ = 0.03</td>
<td>$6.506270 \times 10^{10}$</td>
</tr>
<tr>
<td>15</td>
<td>$\frac{1}{25}$ = 0.04</td>
<td>$2.576910 \times 10^{8}$</td>
</tr>
<tr>
<td>16</td>
<td>$\frac{1}{20}$ = 0.05</td>
<td>$5.92576 \times 10^{6}$</td>
</tr>
<tr>
<td>17</td>
<td>$\frac{3}{50}$ = 0.06</td>
<td>$479437$</td>
</tr>
<tr>
<td>18</td>
<td>$\frac{7}{100}$ = 0.07</td>
<td>$62346.5$</td>
</tr>
<tr>
<td>19</td>
<td>$\frac{7}{25}$ = 0.08</td>
<td>$12234.5$</td>
</tr>
<tr>
<td>20</td>
<td>$\frac{9}{100}$ = 0.09</td>
<td>$4547.64$</td>
</tr>
<tr>
<td>21</td>
<td>$\frac{1}{10}$ = 0.1</td>
<td>$118.104$</td>
</tr>
<tr>
<td>22</td>
<td>$\frac{3}{20}$ = 0.15</td>
<td>$28.2727$</td>
</tr>
<tr>
<td>23</td>
<td>$\frac{1}{5}$ = 0.2</td>
<td>$11.9817$</td>
</tr>
<tr>
<td>24</td>
<td>$\frac{1}{4}$ = 0.25</td>
<td>$7.41419$</td>
</tr>
</tbody>
</table>

Table 12. Values of $B'_2 (\theta_l, \theta_{l+1})$

is 1134, given by the total number of divisions from the third column of Table 13. This is slightly misleading as the last division of $(r_a, r_b) = (0.0861, 0.08622)$ into 1000 intervals of equal length leads to a number of cases where $\mathcal{R}_{10}(\theta_l, \theta_{l+1})$ is the empty set. In other words, $y = \tan(\theta_l)x$ will
lie above $R_{10}$ for $\theta_i \approx 0.08622$. These values of $l$ are to be ignored. What is significant here in fact is that the last $\theta_{i+1}$ considered satisfies $y = \tan(\theta_{i+1})x$ is above $R_{10}$. This is the case due to the value of $R_{10}$ in Table 11.

As suggested by Table 13, for $b \geq 3$, we want the gaps between consecutive $r_i$ considered to become smaller when $B'_b(\theta_i, \theta_{i+1})$ is near the minimum value obtained (in the last column). Apriori, we did not know where the minimum occurs, so we revised the number of divisions (ending with the indicated values in the third column) to be larger until the minimum value of $B'_b(\theta_i, \theta_{i+1})$ was accurate to the first few digits shown.

For a fixed $l \in \{1, 2, \ldots, m - 1\}$, we now show how to obtain a value for $B'_b(\theta_i, \theta_{i+1})$. We have already shown how to find a verifiable lower bound $x'_i$ for the left-most point $(x, y)$ on the intersection of the line $y = \tan(\theta_i)x$ and $R_b$. This was done using a Sturm sequence for a polynomial in $\mathbb{Q}[x]$.

Let

$$
\alpha = x_0 + iy_0 = re^{i\theta} \quad \text{where} \quad (x_0, y_0) \in R_b(\theta_i, \theta_{i+1}).
$$

We will show that both $x_0 \geq x'_i$ and $y_0 \geq \tan(\theta_i)x'_i$. We begin with the former. By way of contradiction, assume that $x_0 < x'_i$. Let $(x_1, y_1)$ be the point where $y = \tan(\theta)x$ intersects $R_b$ with $x_1$ being minimal. Therefore, $(x_1, y_1)$ lies on the boundary of $R_b$, and, by Lemma 3.1, we have that $y_1 = \rho_b(x_1)$. Also, $x_1 \leq x_0 < x'_i$ and, by Lemma 3.1 part (i), $b - a_0 \leq x_1 \leq b + a_1$ where $a_0$ and $a_1$ are given in Table 6. By Lemma 3.1 parts (iii) and (iv), the function $\rho_0(x) = \rho_b(x) - r_lx$ is a continuous function on $I_b = [b - a_0, b + a_1]$ such that $\rho_0(b - a_0) < 0$. However, since $(x_1, y_1) \in R_b(\theta_i, \theta_{i+1})$, it lies above the line $y = \tan(\theta_i)x$. This gives us that

$$
\rho_b(x_1) = y_1 = \tan(\theta)x_1 \geq \tan(\theta_i)x_1 = r_lx_1,
$$

<table>
<thead>
<tr>
<th>$r_a = \tan(\theta_a)$</th>
<th>$r_b = \tan(\theta_b)$</th>
<th># of Divisions</th>
<th>$B'_{10}(\theta_a, \theta_b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.001</td>
<td>1</td>
<td>$1.88 \times 10^{483}$</td>
</tr>
<tr>
<td>0.001</td>
<td>0.002</td>
<td>2</td>
<td>$1.35945 \times 10^{1452}$</td>
</tr>
<tr>
<td>0.002</td>
<td>0.01</td>
<td>8</td>
<td>$9.47832 \times 10^{288}$</td>
</tr>
<tr>
<td>0.01</td>
<td>0.02</td>
<td>2</td>
<td>$5.96751 \times 10^{143}$</td>
</tr>
<tr>
<td>0.02</td>
<td>0.08</td>
<td>6</td>
<td>$1.33634 \times 10^{36}$</td>
</tr>
<tr>
<td>0.08</td>
<td>0.085</td>
<td>5</td>
<td>$3.38637 \times 10^{35}$</td>
</tr>
<tr>
<td>0.085</td>
<td>0.086</td>
<td>100</td>
<td>$2.83670 \times 10^{35}$</td>
</tr>
<tr>
<td>0.086</td>
<td>0.0861</td>
<td>10</td>
<td>$2.75920 \times 10^{35}$</td>
</tr>
<tr>
<td>0.0861</td>
<td>0.08622</td>
<td>1000</td>
<td>$2.74964 \times 10^{35}$</td>
</tr>
</tbody>
</table>

Table 13. Values of $B'_{10}(\theta_i, \theta_{i+1})$
so \( \rho_0(x_1) \geq 0 \). By the Intermediate Value Theorem, there exists a \( u \in [b - a_0, x_1] \) such that \( \rho_0(u) = 0 \). Thus, \( \rho_0(u) = r_1u \), which gives us that \( P_b(u, r_1u) = 0 \). Since
\[
u \leq x_1 < x_1',
\]
we obtain a contradiction to the definition of \( x_1' \). Therefore, \( x_0 \geq x_1' \). To show that \( y_0 \geq \tan (\theta_t) x_1' \), we simply observe now that
\[
y_0 = \tan (\theta) x_0 \geq \tan (\theta_t) x_0 \geq \tan (\theta_t) x_1'.
\]

To get a value for \( B_b'(\theta_t, \theta_{t+1}) \), we used 100 digit approximations in Maple 17 to perform the calculations indicated below. Further details can be found in [4]. We let \( L_l \) be a lower bound approximation of \( \sec (\theta_t) x_1' \), so that, for any \( \alpha = r e^{i\theta} \) as in (5.3), we have that
\[
r = \sqrt{x_0^2 + y_0^2} \geq \sqrt{1 + \tan^2 (\theta_t)} x_1' \geq L_l.
\]
Now, for every \( l \in \{1, 2, \ldots, m - 1\} \), we let \( k_1 = k_1(l) \) be the largest integer \( \leq \pi/\theta_{t+1} \). We define
\[
k_2 = k_2(l) = \begin{cases} k_1 - 1 & \text{if } k_1 - 1 \geq (\pi/2\theta_t) + 10^{-10} \\ k_1 & \text{otherwise.} \end{cases}
\]
Notably, these values depend on the values for \( r_l \) and \( \theta_t \) chosen earlier. In every case, for our choices of \( r_t \) and \( \theta_t \), the inequalities
\[
\frac{\pi}{2\theta_t} + 10^{-10} \leq k_2 \leq k_1 \leq \frac{\pi}{\theta_{t+1}} - 10^{-10}
\]
held. The specific choice of \( 10^{-10} \) is not significant here or later below, but it provides us with some measure of how much accuracy was needed for our computations. For each \( \theta \in [\theta_t, \theta_{t+1}] \), we are able to conclude that
\[
\frac{\pi}{2\theta} \leq \frac{\pi}{2\theta_t} < k_2 \leq k_1 < \frac{\pi}{\theta_{t+1}} \leq \frac{\pi}{\theta}.
\]
Hence, in each case, \( k_1 \) and \( k_2 \) are in the interval \( (\pi/(2\theta), \pi/\theta) \).

For each \( b \) and \( l \), we compute \( c(k_1) \) and \( c(k_2) \) such that
\[
(5.4) \quad \cot (\pi - k_j \theta) \leq \cot (\pi - k_j \theta_{t+1}) \leq c(k_j) - 10^{-10} \quad \text{for } j \in \{1, 2\}.
\]
From the above, Lemma 5.1 now allows us to take
\[
B_b'(\theta_t, \theta_{t+1}) = \max \left\{ \frac{L_{k_1}^1 (L_t - 1)}{1 + c(k_1)}, \frac{L_{k_2}^2 (L_t - 1)}{1 + c(k_2)} \right\}.
\]
These bounds, combined with (5.1) and (5.2), give us the lower bound of \( B_{b(a_n)} \) for at least one of the coefficients of \( f(x) \), where \( B_b \) is as listed in Table 10. Corollary 5.2 now follows. \( \square \)
Before leaving this section, we note that a certain precaution had to be made in (5.4) that is connected to an irrationality result. What happens if our choices for \( \theta_{l+1} \) and \( k_j \) cause the expression \( \cot(\pi - k_1 \theta_{l+1}) \) to be undefined? This in fact can happen. Observe that \( k_1 = \lfloor \pi/\theta_{l+1} \rfloor \). The expression \( \cot(\pi - k_1 \theta_{l+1}) \) is undefined precisely when \( \pi/\theta_{l+1} \in \mathbb{Z} \). If this happens, then \( \theta_{l+1} \) is a rational multiple of \( \pi \). Recall that \( r_{l+1} = \tan(\theta_{l+1}) \) is also rational. The only rational values of the form \( \tan(u\pi) \) with \( u \in \mathbb{Q} \) are 0 and \( \pm 1 \) (cf. Corollary 3.12 in [10]). Thus, for our set-up where \( 0 < \theta_{l+1} < \pi/2 \), we only need avoid \( r_{l+1} = 1 \). Since \( R_b \) is an upper bound on \( r_{l+1} = \tan(\theta_{l+1}) \), we deduce from Table 11 that the possibility of \( r_{l+1} = 1 \) only occurs for \( b = 2 \). This explains the choice of \( r_{47} \) and \( r_{48} \) in Table 12, where we avoided using the rational number 1 for a value of \( r_{l} \).

6. Bounds Based on Recursive Relations

We will now examine another method to bound the coefficients of \( f(x) \) that is motivated by Corollary 5.2. In the case that \( f(x) \) is divisible by one of the quadratics \( \Phi_3(x - b) \), \( \Phi_4(x - b) \) and \( \Phi_6(x - b) \), we find sharp lower bounds for the maximum coefficient of \( f(x) \). The bound that we find will depend on our choice of \( b \) and the quadratic.

As much of this section is based on the work in [7] for \( b = 10 \), we give enough background from there to describe our work for \( b \in [2, 20] \) but refer to [7] for the details of the arguments.

Fix positive integers \( A \) and \( B \). Let \( b_j \) be integers such that

\[
(b_0 x^s + b_1 x^{s-1} + \cdots + b_{s-1} x + b_s) (x^2 - Ax + B)
\]

(6.1)

is a polynomial of degree \( s + 2 \) with non-negative coefficients. We will want \( A \) and \( B \) to be chosen so that the quadratic on the right is one of \( \Phi_3(x - b) \), \( \Phi_4(x - b) \) and \( \Phi_6(x - b) \). With \( f(x) = g(x) h(x) \) as before and \( g(x) \) being the quadratic, we view \( h(x) \) as the polynomial factor on the left in (6.1) and further \( n = \deg f(x) = s + 2 \). The choice of \( b_j \) as the coefficient of \( x^{s-j} \) will help us view the \( b_j \) as forming a sequence and be more appropriate for the arguments that follow. If (6.1) is expanded, we obtain \( f(x) \) so that the resulting coefficients are all non-negative.

We define \( b_j = 0 \) for all \( j < 0 \) and all \( j > s \). Since the coefficients of \( f(x) \) are all non-negative, we deduce that

\[
b_0 \geq 1 \quad \text{and} \quad b_j \geq Ab_{j-1} - Bb_{j-2} \quad \text{for all } j \in \mathbb{Z}.
\]
Define
\begin{equation}
\beta_j = \begin{cases} 
0 & \text{if } j < 0 \\
1 & \text{if } j = 0 \\
A\beta_{j-1} - B\beta_{j-2} & \text{if } j \geq 1,
\end{cases}
\end{equation}
so the \( \beta_j \) satisfy a recursive relation for \( j \geq 0 \). In particular, \( \beta_1 = A \) and \( \beta_2 = A^2 - B \). For each \( A \) and \( B \) corresponding to a quadratic \( x^2 - Ax + B \) equal to one of \( \Phi_3(x - b) \), \( \Phi_4(x - b) \) and \( \Phi_6(x - b) \) for some \( b \in [2, 20] \), the values of \( \beta_j \) vary in sign as \( j \) increases. Let \( J \) be a positive integer for which
\begin{equation}
\beta_j > 0 \quad \text{for } 0 \leq j \leq J.
\end{equation}
As shown in [7], we have
\begin{equation}
b_j \geq \beta_j b_0 \quad \text{for all integers } j \leq J + 1.
\end{equation}
Although it is natural to consider \( J \) maximal satisfying (6.4) as in [7], what we want for our purposes is the least \( J \) for which \( \beta_{J+1} < \beta_J \). In [7], these notions are equivalent; but in general, they are not. Table 14 and Table 15 show the \( A, B, J \) and \( \beta_J \) for \( b \in [2, 20] \). Note that
\[
\beta_J = \max_{0 \leq j \leq J} \{ \beta_j \}.
\]
Let
\[
U = \max_{j \geq 0} \{ b_j \} \quad \text{and} \quad L = \min_{j \geq 0} \{ b_j \}.
\]
Since \( b_j = 0 \) for \( j > s \), we have the trivial bound \( L \leq 0 \). From (6.5), we obtain \( U \geq \beta_J b_0 \).

We are interested in \( A \) and \( B \) such that \( f(x) \) is divisible by \( x^2 - Ax + B \). We view \( A \) and \( B \) as fixed. We want \( f(x) \) to have non-negative integer coefficients but with the largest coefficient as small as possible. Let \( M = M(A, B) \) be the maximum coefficient for such an \( f(x) \). For this definition, we do not require that \( f(b) \) is prime. Thus, if \( f_0(x) \in \mathbb{Z}[x] \) has non-negative integer coefficients and is divisible by \( x^2 - Ax + B \), then \( f_0(x) \) has a coefficient that is \( \geq M \).

We now describe important inequalities obtained in [7]. Let \( \ell \in \mathbb{Z}^+ \). Define \( \mu_0, \mu_1, \ldots, \mu_{\ell-1} \) to be the solution to the matrix equation
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**Table 14.** Values of $\beta_J$ for bases $2 \leq b \leq 12$
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Table 15. Values of $\beta_J$ for bases $13 \leq b \leq 20$
The above corresponds to a system of \( \ell \) equations in the \( \ell \) unknowns \( \mu_j \) where \( 0 \leq j \leq \ell - 1 \). The system of equations depends only on \( A, B \) and \( \ell \). Ideally, we want to know a unique solution to this system of equations exists and each \( \mu_j \in [0, 1] \). For each choice of \( A, B \) and \( \ell \) we use, this can be verified with a direct computation. We suppose then this is the case.

We set

\[
(6.6) \quad u = \mu_0 B, \quad v = \mu_{\ell - 2} - \mu_{\ell - 1} A \quad \text{and} \quad w = \mu_{\ell - 1}.
\]

Then [7] establishes that

\[
(6.7) \quad M \geq \frac{u^2 - (v + w)^2}{u} \cdot U \geq \frac{u^2 - (v + w)^2}{u} \cdot U
\]

\[
\geq \frac{u^2 - (v + w)^2}{u} \cdot \beta_j b_0 \geq \frac{(u^2 - (v + w)^2)\beta_j}{u}
\]

and

\[
(6.8) \quad 0 \leq -L \leq \frac{v + w}{u^2 - (v + w)^2} \cdot M.
\]

The inequalities in (6.7) and (6.8) can be used to estimate \( L \) and \( U \), respectively. We also use (6.7) to find a lower bound for \( M(A, B) \) that is exactly or is close to best possible. With some additional work, as we shall see, we can determine the exact value of \( M(A, B) \). Note that the variables in (6.7) and (6.8) all depend on \( b, A \) and \( B \), and in addition \( u, v \) and \( w \) (as given in (6.6)) depend on \( \ell \). For \( \ell \), we will choose \( \ell = J + 1 \) where \( J \) is given in Table 14 and Table 15.

As an example of the use of (6.7), we can obtain an immediate improvement on Corollary 5.2. Take \( b = 4, A = 9 \) and \( B = 21 \). Computing \( \mu_0, \mu_1, \ldots, \mu_\ell \) with \( \ell = 16 \), we check that the \( \mu_j \) are in \([0, 1]\) and compute \( u, v \) and \( w \) using (6.6). Denoting \( a_n \) as the leading coefficient of \( f(x) \) as in Corollary 5.2, we have \( b_0 = a_n \). Table 14 gives us a lower bound \( b_0 \beta_{15} = a_n \beta_{15} \).
for $U = U(9, 21)$. From (6.7), we see that

$$M = M(9, 21) \geq \frac{u^2 - (v + w)^2}{u} \cdot U \geq 5.6446 \times 10^{10} a_n.$$  

This implies that any polynomial $f(x)$ with non-negative coefficients and leading coefficient $a_n$ that is divisible by $x^2 - 9x + 21$ must have a coefficient as large as $5.6446 \cdot 10^{10} a_n$. From Corollary 5.2, we see that if $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}$ is such that $f(4)$ is prime and

$$0 \leq a_j \leq 5.8802 \cdot 10^7 a_n \quad \text{for } 0 \leq j \leq n,$$

then either $f(x)$ is irreducible or $f(x)$ is divisible by $\Phi_3(x-4) = x^2 - 7x + 13$ or $\Phi_4(x-4) = x^2 - 8x + 17$. Repeating the analogous calculations for bases $2 \leq b \leq 20$, with the aid of Corollary 5.2, we can deduce the following.

**Corollary 6.1** (Improvement of Corollary 5.2). Fix an integer $b$ with $b \geq 2$. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be such that $a_j \geq 0$ for each $j$ and $f(b)$ is prime. If

$$0 \leq a_j \leq B_b a_n \quad \text{for } 0 \leq j \leq n - 1 \quad \text{with } B_b \text{ as in Table 10},$$

then either $f(x)$ is irreducible or $f(x)$ is divisible by at least one of $\Phi_3(x-b)$ and $\Phi_4(x-b)$.

Similarly, for each $b \in [2, 20]$, we can apply (6.7) to find a lower bound for $M(A, B)$ in the case that $g(x) = x^2 - Ax + B$ is one of $\Phi_3(x-b)$ and $\Phi_4(x-b)$. Table 16 and Table 17 lists $b, A, B$, and a lower bound for $M(A, B)$ obtained from our computations. To clarify, these lower bounds are simply $(u^2 - (v + w)^2)\beta_j/u$ as given in (6.7), where again we take $\ell = J + 1$ and we use (6.6) to compute $u, v$ and $w$.

Before proceeding, we note that we have finished establishing the case $b = 2$ of Theorem 1.1. In other words, we can now deduce that if $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $0 \leq a_j \leq 7$ for each $j$ and $f(2)$ prime, then $f(x)$ is irreducible. For $b \in \{3, 4, 5, 6, 7, 14\}$, the bounds $M(A, B)$ come particularly close to what we want. The bounds for $M(A, B)$ establish that $M_1(b)$ can be taken to be one less than what appears in Table 1. In other words, for these $b$, we can now deduce that if $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $0 \leq a_j \leq M_1(b) - 1$ for each $j$ and $f(b)$ prime, then $f(x)$ is irreducible. As we shall see, it is possible for $f(x)$ to have all its coefficients in $[0, M_1(b)]$ with $f(x)$ divisible by $x^2 - Ax + B$. Even though this quadratic has the value 1 at $x = b$, we will see that such an $f(x)$ cannot satisfy $f(b)$ is prime.
Table 16. Lower bound on $M(A, B)$ for $3 \leq b \leq 9$

7. A Sharp Bound for $M(A, B)$

We are now ready to complete the proof of Theorem 1.1. At the end of the previous section, we noted that the case $b = 2$ is complete. For a fixed $b \in [3, 20]$, we are interested in the case that $f(x) = g(x)h(x)$, where $g(x) = x^2 - Ax + B$ is one of $\Phi_3(x - b)$ and $\Phi_4(x - b)$, $h(x)$ has a positive leading coefficient that we have denoted by $b_0$, and also where $f(x)$ has maximal coefficient equal to $M(A, B)$.

We view $A$ and $B$ as fixed. It is worth recalling that $M = M(A, B)$ is the maximal coefficient of $f(x)$ where $f(x)$ is as above but with this maximal coefficient as small as possible. Recall also that we did not require that $f(b)$ is prime in the definition of $M$.

To finish the proof of Theorem 1.1, one checks that it suffices to show both of the following:

(A) The value of $M(A, B) = (1 - A + B) \cdot \beta_J$ for each appropriate choice of $(A, B)$ as shown in Table 14 and Table 15.

(B) If the maximal coefficient of $f(x)$ equals $M$, then $f(b)$ is composite.

Note that in (B), we are supposing as indicated above that $f(x)$ is divisible by $x^2 - Ax + B$. For example, take $b = 8$. Then (A) implies

$$M(16, 65) = 75005556404194608192050$$
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<td>8267439025097901738248191414518610393726802935783728327213629</td>
</tr>
<tr>
<td>17</td>
<td>33</td>
<td>273</td>
<td>977132741058008206920454481120320172727369703845254509827603531966849566</td>
</tr>
<tr>
<td>17</td>
<td>34</td>
<td>290</td>
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</tr>
<tr>
<td>18</td>
<td>35</td>
<td>307</td>
<td>184392431209125593422770054628167938883105685612493543792760301014308216264410882</td>
</tr>
<tr>
<td>18</td>
<td>36</td>
<td>325</td>
<td>21007537854440048721903258296068360516321923712002100068037499036</td>
</tr>
<tr>
<td>19</td>
<td>37</td>
<td>343</td>
<td>2264375758043842756349474421591867656748267691575389195816616747855897250981739624952723</td>
</tr>
<tr>
<td>19</td>
<td>38</td>
<td>362</td>
<td>386253686558080529726943593016202275768222522500247254369696128549408630374397</td>
</tr>
<tr>
<td>20</td>
<td>39</td>
<td>381</td>
<td>296443023675250265371195355803167884005779187086847598894287680701297351967464608428822340</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>401</td>
<td>7965097815841643900684276577174036821605756035173863133380627982979718588470528878</td>
</tr>
</tbody>
</table>

Table 17. Lower bound on $M(A, B)$ for $10 \leq b \leq 20$
and

\[ M(15, 57) = 945987466487208056191224. \]

These are respectively the values of \( M_1(8) \) and \( M_2(8) \) given in Table 1 and Table 2. Corollary 5.2 implies that if \( f(x) = \sum_{j=0}^{\infty} a_j x^j \in \mathbb{Z}[x] \) with \( 0 \leq a_j \leq M(15, 57) \) for each \( j \) and \( f(x) \) is reducible, then \( f(x) \) is divisible by either \( \Phi_3(x - 8) \) or \( \Phi_4(x - 8) \). It follows that if \( 0 \leq a_j \leq M(16, 65) = M_1(8) \), then either \( f(x) \) is irreducible or divisible by \( \Phi_4(x - 8) \). From (B), if also \( f(8) \) is prime, then \( f(x) \) cannot be divisible by \( \Phi_4(x - 8) = x^2 - 16x + 65 \). Therefore, the conditions \( f(8) \) is prime and \( 0 \leq a_j \leq M_1(8) \) in Theorem 1.1 imply \( f(x) \) is irreducible. Similarly, the conditions \( f(8) \) is prime and \( 0 \leq a_j \leq M_2(8) \) in Theorem 1.1 imply either \( f(x) \) is irreducible or \( f(x) \) is divisible by \( \Phi_4(x - 8) \). A similar argument holds for each \( b \in [3, 20] \).

We begin with establishing (A). We suppose at first that

\[ M(A, B) \leq (1 - A + B) \cdot \beta_J. \tag{7.1} \]

Observe that we will want eventually to obtain a contradiction if strict inequality holds in (7.1), but there will be a significance to seeing what the inequality as written in (7.1) gives us. We are interested in the case that \( x^2 - Ax + B \) is one of \( \Phi_3(x - b) \) and \( \Phi_4(x - b) \).

From (6.7) and (7.1), we have

\[ b_0 \beta_J \leq U(A, B) \leq \frac{u M(A, B)}{u^2 - (v + w)^2} \leq \frac{u (1 - A + B) \cdot \beta_J}{u^2 - (v + w)^2}. \tag{7.2} \]

We compute the left-most and right-most sides of (7.2), based on \( u, v \) and \( w \) from (6.6) with \( \ell = J + 1 \) as before, on \( b \in [3, 20] \) and on \( x^2 - Ax + B \) being one of \( \Phi_3(x - b) \) and \( \Phi_4(x - b) \). In all cases, (7.2) gives a contradiction if \( b_0 \geq 2 \), so that we only consider now the possibility that \( h(x) \) is monic. Setting \( b_0 = 1 \) in (7.2), the same computations above lead to \( U = \beta_J \). In other words,

\[ \beta_J = \left\lfloor \frac{u (1 - A + B) \cdot \beta_J}{u^2 - (v + w)^2} \right\rfloor \]

for all \( b \in [3, 20] \) and \( x^2 - Ax + B \) equal to one of \( \Phi_3(x - b) \) and \( \Phi_4(x - b) \).

Using (7.1) with \( u, v \) and \( w \) as before leads to

\[ \frac{v + w}{u^2 - (v + w)^2} \cdot M \in (0, 1) \]

for each \( b \in [3, 20] \) and pair \( (A, B) \). Hence, (6.8) implies that \( L = 0 \).

Thus, we have established (7.1) implies that \( h(x) \) is monic, the largest coefficient of \( h(x) \) corresponds to the value of \( \beta_J \) as indicated in Table 14 and Table 15, and all of the coefficients of \( h(x) \) are non-negative.
The approach given in [7] for \( b = 10 \) follows through for general \( b \) directly at this point to give us more information about the structure of \( h(x) \), based on the information just obtained about \( h(x) \). Following the arguments there, still under the assumption of (7.1), we deduce that \( h(x) \) can be written as a sum over some non-negative integers \( k \) of polynomials which are \( x^k \) times (7.3)

\[
(\beta_0 x^J + \beta_1 x^{J-1} + \cdots + \beta_j) x^{J+1} + \left( x^{J+1} + x^J + \cdots + x^1 \right) \beta_j
\]

\[
+ (\beta_j - \beta_0) x^{J-1} + (\beta_j - \beta_1) x^{J-2} + \cdots + (\beta_j - \beta_{j-1}),
\]

where \( t' = t'(k) \) is a non-negative integer. The \( k \) cannot be arbitrary. There should be no overlapping terms for different \( k \), and the coefficient of \( x^{k-1} \) in \( h(x) \) should be 0 for each \( k \).

We are ready to prove (A). Assume now that strict inequality holds in (7.1). For \( b \in [3, 20] \), we see that \( J \geq 7 \) in Table 14 and Table 15. Observe that, since \( f(x) = (x^2 - Ax + B)h(x) \) with \( h(x) \) as above, \( f(x) \) has a coefficient equal to

\[
(\beta_j - \beta_1) - A(\beta_j - \beta_0) + B\beta_j = (1 - A + B)\beta_j - \beta_1 + A\beta_0
\]

\[
= (1 - A + B)\beta_j,
\]

corresponding to the coefficient of \( x^j \) when the expression in (7.3) is multiplied by \( x^2 - Ax + B \). This contradicts our assumption.

Thus far, we have shown that \( M(A, B) \geq (1 - A + B) \beta_j \). On the other hand, we know the form \( h(x) \) must have if \( M(A, B) = (1 - A + B) \beta_j \). Motivated by (7.3) with \( t' = 0 \), we consider

\[
h_0(x) = \beta_0 x^{2J} + \beta_1 x^{2J-1} + \cdots + \beta_J x^J
\]

\[
+ (\beta_j - \beta_0) x^{J-1} + (\beta_j - \beta_1) x^{J-2} + \cdots + (\beta_j - \beta_{j-1}).
\]

The recursive definition of \( \beta_j \) now implies that

\[
(x^2 - Ax + B) h_0(x)
\]

\[
= x^{2J+2} + ((1 - A)\beta_j + B\beta_{j-1} - 1) x^{J+1}
\]

\[
+ (1 - A + B) \beta_j x^J + \cdots + (1 - A + B) \beta_j x^2
\]

\[
+ ((B - A) \beta_j + A\beta_{j-1} - B \beta_{j-2}) x + B (\beta_j - \beta_{j-1}).
\]

Note that the coefficient of \( x \) here can be rewritten as \( (1 - A + B) \beta_j \). Furthermore, the constant term of \( (x^2 - Ax + B) h_0(x) \) can be rewritten as

\[
(1 - A + B) \beta_j - \beta_j + \beta_{j+1}.
\]
Recalling the definition of $J$ gives $\beta_{J-1} \leq \beta_J$ and $\beta_{J+1} < \beta_J$, we see that the maximal coefficient of $(x^2 - Ax + B) h_0(x)$ is $(1 - A + B) \beta_J$. The definition of $M(A, B)$ now implies the equality given in (A).

Now, we prove (B). The approach here differs from that given in [7] and necessarily has to be different for some values of $b \in [3, 20]$. By (A), we know $M(A, B) = (1 - A + B) \beta_J$, so that $f(x) = (x^2 - Ax + B) h(x)$ where $h(x)$ is a sum over some non-negative integers $k$ of polynomials which are $x^k$ times polynomials of the form (7.3). We refer to the polynomial in (7.3) as part of $h(x)$. We begin by showing that with $A, B$ and $J$ fixed, but $t'$ arbitrary, each part of $h(x)$ is divisible by

$$h_1(x) = \sum_{j=0}^{J} (\beta_{J-j} - \beta_{J-j-1}) x^j,$$

where we recall here that $\beta_{-1} = 0$. From this definition of $h_1(x)$, we have

$$\sum_{j=0}^{J} \beta_{J-j} x^j \equiv \sum_{j=0}^{J} \beta_{J-j-1} x^j \equiv \sum_{j=1}^{J} \beta_{J-j} x^{j-1} \pmod{h_1(x)}.$$

We deduce that the polynomial given in (7.3) is

$$\left( \sum_{j=0}^{J} \beta_{J-j} x^j \right)^{x^{j+t'}} + \left( \sum_{j=0}^{J} x^j \right) \beta_J - \sum_{j=1}^{J} \beta_{J-j} x^{j-1}$$

$$\equiv \left( \sum_{j=1}^{J} \beta_{J-j} x^{j-1} \right)^{x^{j+t'}} + \left( \sum_{j=0}^{J} x^j \right) \beta_J - \sum_{j=1}^{J} \beta_{J-j} x^{j-1}$$

$$\equiv \left( \sum_{j=0}^{J} \beta_{J-j} x^j \right)^{x^{j+t'-1}} + \left( \sum_{j=0}^{J} x^j \right) \beta_J - \sum_{j=1}^{J} \beta_{J-j} x^{j-1}$$

$$\equiv \left( \sum_{j=0}^{J} \beta_{J-j} x^j \right)^{x^{j+t'-2}} + \left( \sum_{j=0}^{J} x^j \right) \beta_J - \sum_{j=1}^{J} \beta_{J-j} x^{j-1}$$

$$\vdots$$

$$\equiv \sum_{j=0}^{J} \beta_{J-j} x^j - \sum_{j=1}^{J} \beta_{J-j} x^{j-1} \equiv 0 \pmod{h_1(x)}.$$

Thus, we obtain that each part of $h(x)$ and, therefore, $h(x)$ itself is divisible by $h_1(x)$. Using that $h(x)$ consists of at least one part as in (7.3) with $t' \geq 0$ and $J \geq 1$, we deduce that

$$h(b) \geq (\beta_0 b^t + \beta_1 b^{t-1} + \cdots + \beta_J) b^t > \beta_0 b^t + \beta_1 b^{t-1} + \cdots + \beta_J > h_1(b) > 1.$$

Hence, $h(b)$ is the integer $h_1(b)$ times an integer that is $> 1$. We deduce that $f(b) = g(b)h(b) = h(b)$ is composite. This finishes the proof of (B).
Recall that this completes our proof of Theorem 1.1, but we are still interested in showing that most of the bounds in Theorem 1.1 are sharp as indicated after the statement of Theorem 1.1.

To establish that bounds are sharp in Theorem 1.1, we find explicit examples of reducible \( f(x) \in \mathbb{Z}[x] \) with non-negative coefficients, with maximal coefficient equal to \((1 - A + B)\beta_J + 1\) and with \( f(b) \) prime. To find explicit examples, we fix an integer \( b \in [3, 20] \), choose the appropriate \( A, B \) and \( J \) using Table 14 and Table 15, and then we take \( h_1(x) \) to be as given in (7.3).

In each case, we set \( t' = 0 \) except for the case \((b, A, B) = (15, 30, 226)\) where we set \( t' = 1 \). With some trial and error, we found a quadratic \( h_2(x) \in \mathbb{Z}[x] \) such that \( h(x) = h_1(x) + h_2(x) \) satisfies the following conditions:

- \( f(x) = (x^2 - Ax + B) h(x) \) has non-negative coefficients,
- \( f(b) \) is prime,
- the largest coefficient of \( f(x) \) is \((1 - A + B)\beta_J + 1\),

where \( \beta_J \) is given in Table 14 or Table 15. So as to save space in the representations of the polynomial examples we found, we indicate \( f(x) \) by only tabulating \( h_2(x) \). Observe that the value of \( h_2(x) \) uniquely determines an \( f(x) \) as described. Table 18 below gives our explicit choices of \( h_2(x) \) to construct \( f(x) \) showing us that the bounds \( M_1(b) \) for \( b \in [3, 20] \) and the bounds \( M_2(b) \) for \( b \in [4, 20] \) given in Theorem 1.1 are sharp.

8. Final Arguments

We finish by supplying a proof of Theorem 4.3 and, in particular, examples justifying the degree bounds in Theorem 4.2 and the coefficient bounds in Theorem 4.3 are sharp. The bounds from Corollary 6.1 imply that we need only consider the case that \( f(x) = g(x)h(x) \) where \( g(x) = x^2 - Ax + B \) is one of \( \Phi_3(x - b) \) and \( \Phi_4(x - b) \) and where \( h(x) \) can be taken in the form of the first factor in (6.1). In particular, (6.1) equals \( f(x) \).

Fix \( b \in [2, 20] \). Let \( f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x] \) such that \( a_j \geq 0 \) for each \( j \) and \( f(b) \) is prime. From (6.5), we have

\[
b_j \geq \beta_j b_0 \quad \text{if} \quad \beta_i > 0 \quad \text{for} \quad 0 \leq i \leq j - 1.
\]

Set

\[
J_0 = J_0(b, A, B) = \begin{cases} J & \text{if} \ \beta_{J+1} < 0 \\ J + 1 & \text{if} \ \beta_{J+1} \geq 0. \end{cases}
\]

For \((b, A, B) = (2, 3, 3)\), one checks that \( \beta_{J_0} = \beta_{J+1} = 0 \). For all other \((b, A, B)\) under consideration, \( \beta_{J_0} > 0 \). Thus,

\[
\beta_j > 0 \quad \text{for} \quad 0 \leq j \leq J_0, \quad \text{if} \quad (b, A, B) \neq (2, 3, 3) \quad \text{or} \quad j \neq J_0.
\]
Table 18. Examples of \( h_2(x) \) for \( M_1(b) \) and \( M_2(b) \)

<table>
<thead>
<tr>
<th>( b )</th>
<th>( h_2(x) ) for ( M_1(b) )</th>
<th>( h_2(x) ) for ( M_2(b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( x^2 + 5x + 9 )</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>( x^2 + 5x + 10 )</td>
<td>( x^2 + 8x + 40 )</td>
</tr>
<tr>
<td>5</td>
<td>( x^2 + 7x + 23 )</td>
<td>( x^2 + 10x + 44 )</td>
</tr>
<tr>
<td>6</td>
<td>( x^2 + 8x + 32 )</td>
<td>( x^2 + 11x + 48 )</td>
</tr>
<tr>
<td>7</td>
<td>( x^2 + 9x + 39 )</td>
<td>( x^2 + 13x + 46 )</td>
</tr>
<tr>
<td>8</td>
<td>( x^2 + 15x + 72 )</td>
<td>( x^2 + 15x + 106 )</td>
</tr>
<tr>
<td>9</td>
<td>( x^2 + 16x + 76 )</td>
<td>( x^2 + 17x + 115 )</td>
</tr>
<tr>
<td>10</td>
<td>( x^2 + 8x + 54 )</td>
<td>( x^2 + 11x + 66 )</td>
</tr>
<tr>
<td>11</td>
<td>( x^2 + 14x + 84 )</td>
<td>( x^2 + 21x + 133 )</td>
</tr>
<tr>
<td>12</td>
<td>( x^2 + 19x + 126 )</td>
<td>( x^2 + 23x + 135 )</td>
</tr>
<tr>
<td>13</td>
<td>( x^2 + 16x + 122 )</td>
<td>( x^2 + 13x + 83 )</td>
</tr>
<tr>
<td>14</td>
<td>( x^2 + 14x + 114 )</td>
<td>( x^2 + 23x + 164 )</td>
</tr>
<tr>
<td>15</td>
<td>( x^2 + 24x + 198 )</td>
<td>( x^2 + 15x + 123 )</td>
</tr>
<tr>
<td>16</td>
<td>( x^2 + 12x + 114 )</td>
<td>( x^2 + 31x + 565 )</td>
</tr>
<tr>
<td>17</td>
<td>( x^2 + 18x + 178 )</td>
<td>( x^2 + 19x + 176 )</td>
</tr>
<tr>
<td>18</td>
<td>( x^2 + 19x + 198 )</td>
<td>( x^2 + 35x + 742 )</td>
</tr>
<tr>
<td>19</td>
<td>( x^2 + 29x + 279 )</td>
<td>( x^2 + 27x + 272 )</td>
</tr>
<tr>
<td>20</td>
<td>( x^2 + 21x + 232 )</td>
<td>( x^2 + 39x + 522 )</td>
</tr>
</tbody>
</table>

For \( (b, A, B) \neq (2, 3, 3) \), we deduce that \( b_j > 0 \) for all \( j \leq J_0 \); in particular, \( s = \deg h \geq J_0 \) and \( \deg f \geq J_0 + 2 \). In the proof of Theorem 4.2, we established \( \deg f \geq J_0 + 2 \) in the case \( (b, A, B) = (2, 3, 3) \). In fact, for \( b \in [2, 20] \), we note that \( J_0 + 1 \) agrees with the values of \( D(b) \) and \( D_1(b) \) given in Table 7. In particular, to justify \( D(b) \) is sharp and to justify the value of \( N_1(b) \) given in Table 8, we will take \( s = J_0 \) and \( \deg f = J_0 + 2 \) with the maximal coefficient of \( f(x) \) as small as possible.

Recall \( b_j \) has been defined for all integers \( j \). We now define

\[
\kappa_j = b_j - Ab_{j-1} + Bb_{j-2} \quad \text{for } j \in \mathbb{Z}.
\]

Observe that \( \kappa_j \geq 0 \) for all \( j \in \mathbb{Z} \). For integers \( u \) and \( t \), we also define

\[
\kappa'(u,t) = \sum_{j=0}^{u} \beta_j \kappa_{t-j}.
\]
Thus, $\kappa'(u, t) = \kappa'(u - 1, t) + \beta_u \kappa_{t-u}$. Recall $\beta_0 = 1$, $\beta_1 = A$ and $\beta_{j+1} = A\beta_j - B\beta_{j-1}$ for $j \geq 1$. Using the definition of $\kappa_j$, we deduce
\[
b_t = \beta_1 b_{t-1} - B\beta_0 b_{t-2} + \kappa'(0, t)
\]
\[
= \beta_1 \left( Ab_{t-2} - B\beta_0 b_{t-3} + \kappa_{t-1} \right) - B\beta_0 b_{t-2} + \kappa'(0, t)
\]
\[
= \beta_2 b_{t-2} - B\beta_1 b_{t-3} + \kappa'(1, t)
\]
\[
= \cdots = \beta_{t-2} b_2 - B\beta_{t-3} b_1 + \kappa'(t - 3, t)
\]
\[
= \beta_{t-1} b_1 - B\beta_{t-2} b_0 + \kappa'(t - 2, t)
\]
\[
= \beta_t b_0 + \kappa'(t - 1, t).
\]

For reference purposes, we summarize the above as
\[
(8.2) \quad b_t = \beta_t b_0 + \kappa'(t - 1, t).
\]

There are two strategies we consider at this point. The first strategy is derived from [7] and applies in most cases. In each strategy, the basic idea is that $h(x)$ should not differ much from
\[
h_3(x) = \beta_0 x^{j_0} + \beta_1 x^{j_0-1} + \cdots + \beta_{j_0-1} x + \beta_{j_0},
\]
where the subscript 3 on the left is used only to avoid conflicts with previous notation. We will tabulate examples of $f(x)$ more efficiently by tabulating instead
\[
h_4(x) = h(x) - h_3(x) = \sum_{j=0}^{j_0} (b_j - \beta_j) x^{j_0-j}.
\]
Thus, $f(x) = (x^2 - Ax + B)(h_3(x) + h_4(x))$, where $A$ and $B$ come from the coefficients of either $\Phi_3(x - b)$ or $\Phi_4(x - b)$ and where $h_3(x)$ is derived directly from the recurrence for $\beta_j$ made explicit in (6.3).

Given the above, the expression $(x^2 - Ax + B)h_3(x)$ can be viewed as an approximation of $f(x)$. The coefficient of $x$ in $(x^2 - Ax + B)h_3(x)$ and the constant term of $(x^2 - Ax + B)h_3(x)$ are
\[
B\beta_{j_0-1} - A\beta_{j_0} \quad \text{and} \quad B\beta_{j_0},
\]
respectively. Strategy I will provide us with the $h_4(x)$ we want in the case that the constant term is at least as large as the coefficient of $x$. Thus, we use Strategy I when
\[
B\beta_{j_0} \geq B\beta_{j_0-1} - A\beta_{j_0}.
\]
Note that, in particular, this inequality holds if $\beta_{j_0} \geq \beta_{j_0-1}$, which is typically the case. Strategy II applies when the above inequality does not hold. This leads to applying Strategy II only in the cases $b \in \{6, 14\}$ (with $g(x)$ either of $\Phi_3(x - b)$ and $\Phi_4(x - b)$) and $(b, A, B) = (2, 3, 3)$. The case
$(b, A, B) = (7, 14, 50)$ is the unique case in our computations where Strategy I applies but $\beta_{J_0} < \beta_{J_0 - 1}$.

The results of applying Strategy I and Strategy II appear in Table 19 and Table 20, respectively. In Table 20, the second column distinguishes whether $\Phi_3(x - b)$ or $\Phi_4(x - b)$ is being used, the value 3 referring to the former and the value 4 to the latter.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$b$ & $h_4(x)$ for $\Phi_3(x - b)$ & $h_4(x)$ for $\Phi_4(x - b)$ \\
\hline
2 & 3 & \\
3 & $x + 8$ & 0 \\
4 & $x + 7$ & $x + 13$ \\
5 & $2x + 28$ & 14 \\
7 & $6x + 95$ & 8 \\
8 & $x + 29$ & $5x + 80$ \\
9 & $6x + 115$ & $4x + 92$ \\
10 & $3x + 60$ & $4x + 90$ \\
11 & 21 & 28 \\
12 & $x + 48$ & $4x + 102$ \\
13 & $x + 62$ & 2 \\
15 & $6x + 192$ & $9x + 279$ \\
16 & $x + 68$ & $4x + 139$ \\
17 & $3x + 100$ & 12 \\
18 & $5x + 211$ & $2x + 113$ \\
19 & $4x + 176$ & 12 \\
20 & $x + 72$ & $5x + 233$ \\
\hline
\end{tabular}
\caption{\textit{h}_4(x) \text{ from Strategy I}}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$b$ & $\Phi$ \\
\hline
2 & 3 & $x + 7$ \\
6 & 3 & 466236134270 \\
6 & 4 & $2x + 13519269991344$ \\
14 & 3 & $2x + 54237181819689662822645558359568793540061708639396290$ \\
14 & 4 & $9x + 190427015436250536820510121014683293286454260001$ \\
\hline
\end{tabular}
\caption{\textit{h}_4(x) \text{ from Strategy II}}
\end{table}

\textbf{Strategy I}. The basic idea here is to focus on the constant term of $f(x)$ as being its largest coefficient. Here, we take $t = J_0$ in (8.2). The constant term
of \( h(x) \) is \( b_{J_0} \), and we view (8.2) as indicating how far this constant term is from \( \beta_{J_0} b_0 \). Note that the constant term of \( f(x) \) is \( B b_{J_0} \). If the maximal coefficient of \( f(x) \) is \( M \), then necessarily \( B b_{J_0} \leq M \) and we deduce

\[
(8.3) \quad \beta_{J_0} b_0 + \sum_{j=0}^{J_0 - 1} \beta_j \kappa_{J_0 - j} = b_{J_0} \leq \frac{M}{B}.
\]

The idea is to choose an upper bound search value \( M' \) for \( M \) that is close to \( B \beta_{J_0} \). We take \( M' = B \beta_{J_0} + M_0' \) where \( M_0' > 0 \) is relatively small (a value \( \leq 95000 \) sufficed for each polynomial we tested but often much smaller values as well). We then seek to determine the polynomials \( f(x) \) with maximal coefficient \( M \in [B \beta_{J_0}, M'] \) that are of the form (6.1). If none exists, we increase the value of \( M_0' \). As long as we find such an \( f(x) \) with \( M_0' \leq \beta_{J_0} \), we know from (8.3) that \( b_0 = 1 \) when \( M \) is minimal. The definition of \( \kappa_0 \) then implies in this case that \( \kappa_0 = 1 \).

From (8.3), we obtain

\[
\beta_{J_0} + \sum_{j=0}^{J_0 - 1} \beta_j \kappa_{J_0 - j} \leq \frac{B \beta_{J_0} + M_0'}{B} = \beta_{J_0} + \frac{M_0'}{B}.
\]

Hence,

\[
(8.4) \quad \sum_{j=0}^{J_0 - 1} \beta_j \kappa_{J_0 - j} \leq \frac{M_0'}{B}.
\]

Since the values of \( \beta_j \) grow quickly as \( j \) increases, if \( M_0' \) is relatively small, then (8.4) forces \( \kappa_{J_0 - j} \) to be 0 unless \( j \) is small. This then allows us to determine a small number of choices for the \( \kappa_j \) and, therefore, a small number of choices of \( b_j \) from (8.1). Thus, we are left with a small number of \( h(x) \) and, hence, \( f(x) \) to consider.

As an example, consider \( b = 7 \) and \( g(x) = \Phi_3(x - 7) = x^2 - 13 x + 43 \). Thus, \( A = 13 \) and \( B = 43 \), and one checks that \( J_0 = J = 22 \). Take \( M_0' = 5000 \). Then (8.4) implies

\[
\sum_{j=0}^{21} \beta_j \kappa_{22 - j} \leq \frac{5000}{43} \leq 116.28.
\]

Given \( \beta_0 < \beta_1 < \cdots < \beta_{22} \approx 6.68 \cdot 10^{17} \) and

\[
\beta_0 = 1, \quad \beta_1 = 13, \quad \beta_2 = 126, \ldots
\]

we deduce \( \kappa_0 = \kappa_1 = \cdots = \kappa_{20} = 0, \kappa_{21} \leq 8, \) and \( \kappa_{22} \leq 116 \). Thus, there are 9 possibilities for \( \kappa_{21} \in [0, 8] \) and 117 choices for \( \kappa_{22} \in [0, 116] \), giving a total of \( 9 \times 117 = 1053 \) choices for the \( \kappa_j \). Each of these leads to a polynomial \( h(x) = \sum_{j=0}^{22} b_j x^j \) using (8.1). These 1053 polynomials \( h(x) \)
include all possibilities for \( h(x) \in \mathbb{Z}[x] \) for which \( f(x) = (x^2 - 13x + 43)h(x) \) is of degree 24 and has non-negative coefficients all bounded above by \( B\beta_{J_0} + 5000 \). We are interested in those \( f(x) \) for which \( f(7) = h(7) \) is prime, and we want the maximal coefficient of such an \( f(x) \) to be as small as possible. A direct check gives that \( \kappa_{21} = 6 \) and \( \kappa_{22} = 17 \) produces such an \( f(x) \).

**Strategy II.** For this approach, we focus on both the coefficient of \( x \) and the constant term of \( f(x) \). These coefficients are

\[
Bb_{J_0-1} - Ab_{J_0} \quad \text{and} \quad Bb_{J_0},
\]

respectively. If the maximal coefficient of \( f(x) \) is \( M \), then a weighted average of these coefficients must also be \( \leq M \). In particular, we deduce that

\[
\frac{B^2}{A+B}b_{J_0-1} = \frac{B}{A+B}(Bb_{J_0-1} - Ab_{J_0}) + \frac{A}{A+B}(Bb_{J_0}) \leq M.
\]

We apply (8.2) with \( t = J_0 - 1 \) to deduce that

\[
\beta_{J_0-1}b_0 + \sum_{j=0}^{J_0-2} \beta_j \kappa_{J_0-1-j} = \beta_{J_0-1}b_0 + \kappa'(J_0-2, J_0-1) = b_{J_0-1} \leq \frac{A+B}{B^2} \cdot M.
\]

We deduce that \( M \geq B^2\beta_{J_0-1}/(A+B) \). We choose an upper bound search value \( M' \) for \( M \) that is close to \( B^2\beta_{J_0-1}/(A+B) \). We take

\[
M' = \frac{B^2}{A+B} \cdot \beta_{J_0-1} + M_0', \quad \text{with} \quad M_0' < \frac{B^2}{A+B} \cdot \beta_{J_0-1}.
\]

The upper bound on \( M_0' \) is considerably larger than we want in general, and this upper bound ensures that \( b_0 = 1 \) and hence, by definition, \( \kappa_0 = 1 \). We deduce now that

\[
\sum_{j=0}^{J_0-2} \beta_j \kappa_{J_0-1-j} \leq \frac{A+B}{B^2} \cdot M_0'.
\]

With \( M_0' \) small, we are able to deduce reasonable upper bounds from (8.5) for every \( \kappa_j \) except \( \kappa_{J_0} \).

The value \( \kappa_{J_0} \) can be very large, and the idea is to find a very close approximation \( \kappa^* \in \mathbb{Z} \) to \( \kappa_{J_0} \) and to use this to narrow down the possibilities for \( \kappa_{J_0} \). The value of \( \kappa^* \) will depend on the values of \( \kappa_0, \kappa_1, \ldots, \kappa_{J_0-1} \). We fix \( \kappa_j \) for \( j \in \{0, 1, \ldots, J_0-1\} \) from the finite collection of possibilities determined by (8.5). By the definition of the \( \kappa_j \), the values of \( b_j \) are determined for \( j \in \{0, 1, \ldots, J_0-1\} \). The idea now is to choose \( \kappa^* \) so that the selection \( \kappa_{J_0} = \kappa^* \) forces the coefficient of \( x \) in \( f(x) \) to be close to the constant term of \( f(x) \). One can check that this leads to

\[
\kappa^* = \left[ Bb_{J_0-2} - Ab_{J_0-1} + \frac{Bb_{J_0-1}}{A+B} + \frac{1}{2} \right].
\]
though the justification of this choice for $\kappa^*$ is not needed to see that it provides us with an estimate that will allow us to determine $\kappa_{J_0}$. We explain this next.

Fixing $\kappa^*$ as above, we show that $\kappa_{J_0}$ must be close to $\kappa^*$. Set $\kappa_{J_0} = \kappa^* + t$.

Thus, we are interested in showing that $|t|$ is not very large. Since the coefficients of $f(x)$ must be $\leq M$, by looking at the coefficient of $x$ in $f(x)$, we deduce that

$$Bb_{J_0-1} - A(\frac{A}{2} - At \leq M \leq M' = \frac{B^2}{A+B} \cdot \beta_{J_0-1} + M'_0,$$ which simplifies to

$$t \geq \frac{B^2}{A(A+B)} \cdot (b_{J_0-1} - \beta_{J_0-1}) - \frac{M'_0}{A} - \frac{1}{2}.$$

By looking at the constant term in $f(x)$, we deduce that

$$B(\frac{A}{2} - At \leq M \leq M' = \frac{B^2}{A+B} \cdot \beta_{J_0-1} + M'_0).$$

Since

$$\kappa^* > Bb_{J_0-2} - A\beta_{J_0-1} - \frac{b_{J_0-1}}{A+B} - \frac{1}{2},$$

we are led to

$$t < -\frac{B}{A+B} \cdot (b_{J_0-1} - \beta_{J_0-1}) + \frac{M'_0}{B} + \frac{1}{2}.$$

Observe that (8.2) implies $b_{J_0-1} - \beta_{J_0-1} \geq 0$. Although not needed, (8.5) also implies $b_{J_0-1} - \beta_{J_0-1}$ is not very large. In particular, we deduce that

$$-\frac{M'_0}{A} - \frac{1}{2} \leq t < \frac{M'_0}{B} + \frac{1}{2}.$$

Given $\kappa_{J_0} = \kappa^* + t$, we are left with only a small number of choices for $\kappa_{J_0}$, and can test for $f(x)$ as in Strategy I.

As an example, we consider $(b, A, B) = (6, 12, 37)$. We take $M'_0 = 200$. One checks that $J_0 = J + 1 = 18$. As is easily checked, then, $0 < \beta_0 < \beta_1 < \cdots < \beta_{J_0-1}$ and $0 < \beta_{J_0} < \beta_{J_0-1}$. Also,

$$\frac{(A+B)M'_0}{B^2} = \frac{49 \cdot 200}{37^2} = 7.1585 \ldots.$$ From (8.5), we deduce $\kappa_0 = \kappa_1 = \cdots = \kappa_{16} = 0$ and $0 \leq \kappa_{17} \leq 7$. We set $b_0 = 1$. For each value of $\kappa_{17} \in [0, 7]$, we use (8.2) to compute the values of
$b_1, b_2, \ldots, b_{17}$, (8.6) to compute $\kappa^*$, and (8.7) and (8.8) to find the bounds for $t$. The choice of $\kappa_{17}$ that leads to the maximal coefficient of an $f(x)$ as small as possible and with $f(6)$ prime is $\kappa_{17} = 2$. This choice of $\kappa_{17}$ gives

$$\kappa^* = 13519269991324 \quad \text{and} \quad -12 \leq t \leq 4.$$

The desired $f(x)$ comes from the choice $t = -4$, where

$$b_{18} = 12b_{17} - 37b_{16} + \kappa^* + t$$

$$= 12b_{17} - 37b_{16} + 13519269991320$$

$$= 16497794651771.$$

Thus,

$$f(x) = (x^2 - 12x + 37)h(x) \quad \text{with} \quad h(x) = b_0x^{18} + b_1x^{17} + \cdots + b_{17}x + b_{18}$$

and with $f(6) = h(6)$ prime. The maximal coefficient of $f(x)$ is

$$610418402115746,$$

corresponding to the coefficient of $x$ in $f(x)$.

A similar use of Strategy II for $(b,A,B) = (6,11,31)$ establishes that the smallest maximal coefficient of an $f(x)$ having non-negative coefficients with $f(x)$ divisible by $\Phi_3(x-6)$ and $f(6)$ prime is 674230217165581. In terms of Theorem 4.3, these examples justify the values of $N_1(6) = 610418402115745$ and $N_2(6) = 674230217165580$ given in Table 8 and Table 9 are sharp.

### 9. Concluding Remarks

Having dealt with the cases $b \in [2,20]$, it is natural to ask what can be said for $b \geq 21$ or $b$ large. In a subsequent paper, we plan to discuss results for general $b \geq 2$, where what we have established in this paper can be combined with analysis for larger $b$ to obtain explicit results for all $b \geq 2$. For example, Theorem 4.2 in combination with an analysis for larger $b$ leads to the following.

**Theorem 9.1.** Let $b$ be an integer $\geq 2$, and let $D = D(b) = \lceil \pi / \tan^{-1}(1/b) \rceil$. Then there are no reducible $f(x) \in \mathbb{Z}[x]$ of degree $\leq D$ having non-negative integer coefficients for which $f(b)$ is prime. Furthermore, for every integer $n > D$, there are infinitely many reducible $f(x) \in \mathbb{Z}[x]$ of degree $n$ having non-negative integer coefficients with $f(b)$ prime.

As indicated early on in this paper, the analysis for smaller $b$ tends to be more difficult. In particular, recall that we have not been able to establish a sharp bound for $M_1(2)$ or $M_2(3)$. We view finding a sharp bound for $M_1(2)$ as a particularly interesting challenge for further investigation.
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