FACTORING SPARSE POLYNOMIALS

Theorem 1 (Schinzel): Let r be a positive integer, and fix non-zero integers a_0, \ldots, a_r . Let

 $F(x_1,\ldots,x_r)=a_rx_r+\cdots+a_1x_1+a_0.$

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Then there exist finite sets S and T of matrices satisfying:

(i) Each matrix in S or T is an r × ρ matrix with integer entries and of rank ρ for some ρ ≤ r.
(ii) The matrices in S and T are computable.

(iii) For every set of positive integers d_1, \ldots, d_r with $d_1 < d_2 < \cdots < d_r$, the non-reciprocal part of $F(x^{d_1}, \ldots, x^{d_r})$ is reducible if and only if there is an $r \times \rho$ matrix N in S and integers v_1, \ldots, v_ρ satisfying

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 $F(x_1, \dots, x_r) = a_r x_r + \dots + a_1 x_1 + a_0$ $F(x^{d_1}, \dots, x^{d_r}) = a_r x^{d_r} + \dots + a_1 x^{d_1} + a_0$ **Theorem 2 (Schinzel):** Let r be a positive integer, and fix non-zero integers a_0, \ldots, a_r . Let

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(iii) For every set of positive integers d_1, \ldots, d_r with $F(x^{d_1}, \ldots, x^{d_r})$ not reciprocal and $d_1 < d_2 < \cdots < d_r$, the *non-cyclotomic* part of $F(x^{d_1}, \ldots, x^{d_r})$ is reducible if and only if there is an $r \times \rho$ matrix N in S and integers v_1, \ldots, v_ρ satisfying

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Theorem: There is an algorithm with the following property: Given a non-reciprocal $f(x) \in \mathbb{Z}[x]$ with N non-zero terms, degree n and height H, the algorithm determines whether f(x) is irreducible in time

$$c(N,H)(\log n)^{c'(N)}$$

where c(N, H) depends only on N and H and c'(N) depends only on N.

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Consider

$$F(x_1,\ldots,x_r)=a_rx_r+\cdots+a_1x_1+a_0$$
 so that

$$F(x^{d_1},...,x^{d_r}) = a_r x^{d_r} + \cdots + a_1 x^{d_1} + a_0.$$

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Begin the algorithm by constructing the finite sets S and T of matrices in Schinzel's Theorem 2.

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Begin the algorithm by constructing the finite sets S and T of matrices in Schinzel's Theorem 2. Observe that S and T depend on F and not on the d_1, \ldots, d_r .

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Begin the algorithm by constructing the finite sets S and T of matrices in Schinzel's Theorem 2. Observe that S and T depend on F and not on the d_1, \ldots, d_r , so this takes running time $\leq c_1(N, H)$.

Next, the algorithm checks each matrix \boldsymbol{N} in \boldsymbol{S} to see if

$$egin{pmatrix} {d_1} \\ {d_2} \\ {ec s} \\ {d_r} \end{pmatrix} = N egin{pmatrix} {v_1} \\ {v_2} \\ {ec s} \\ {ec s}
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for some integers v_1, \ldots, v_{ρ} . In other words, v_1, \ldots, v_{ρ} are unknowns and elementary row operations are done to solve the above system of equations.

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for some integers $v'_1, \ldots, v'_{\rho'}$ by using elementary row operations to solve the system of equations for the v'_j .

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Schinzel's Theorem 2 now indicates to us whether f(x) has a reducible non-cyclotomic part. If so, then we output that f(x) is reducible. If not, we have more work to do.

Recall, we had the following theorem.

Theorem: There is an algorithm that has the following property: given $f(x) = \sum_{j=1}^{N} a_j x^{d_j} \in \mathbb{Z}[x]$ with deg f = n, the algorithm determines whether f(x) has a cyclotomic factor and with running time

$$\ll \expig((2+o(1))\sqrt{N/\log N}(\log N+\log\log n)ig) \ imes \log(H+1)$$

as N tends to infinity, where $H = \max_{1 \le j \le N} \{|a_j|\}$.

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We'll come back to this.

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Algorithm Continued: If the non-cyclotomic part of f(x) is irreducible, then use the algorithm in the above theorem.

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Algorithm Continued: If the non-cyclotomic part of f(x) is irreducible, then use the algorithm in the above theorem. This completes the proof of the theorem.

A CURIOUS CONNECTION WITH THE ODD COVERING PROBLEM

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A covering of the integers is a system of congruences

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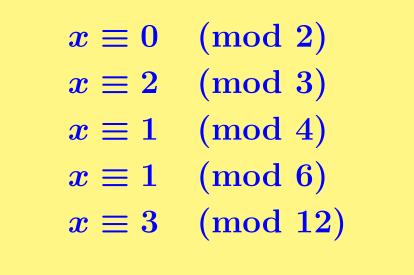
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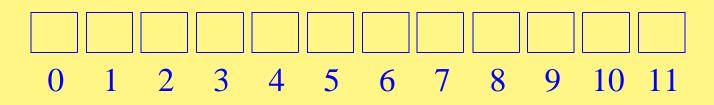
Example 1:

<i>x</i> =	≡ 0	(mod	2)
$oldsymbol{x}$ =	≡ 1	(mod	2)

Example 2:

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Does there exist an "odd covering" of the integers, a covering consisting of distinct odd moduli > 1?

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Selfridge: \$2000 (for explicit example)

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Answer: Nobody knows.

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Comment: For each *n*, the above polynomial is divisible by at least one of

 $\Phi_k(x)$ where $k \in \{2, 3, 4, 6, 12\}$.

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Theorem. There exists an $f(x) \in \mathbb{Z}[x]$ with non-negative coefficients such that $f(x)x^n + 4$ is reducible for all non-negative integers n.

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Theorem. There exists an $f(x) \in \mathbb{Z}[x]$ with non-negative coefficients such that $f(x)x^n + 4$ is reducible for all non-negative integers n.

Comment: For each n, the first polynomial is divisible by at least one $\Phi_k(x)$ where k divides 12.

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Theorem. There exists an $f(x) \in \mathbb{Z}[x]$ with non-negative coefficients such that $f(x)x^n + 4$ is reducible for all non-negative integers n.

Comment: For each n, the second polynomial is divisible by at least one $\Phi_k(x)$ where k divides some integer Nhaving more than 10^{17} digits.

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 $2^{436750334086348800} 3^{41} 5^{31} 7^{37} 11^{29} 13^{23} 17^{16} 19^{18} 23^{23} 29^{29} 31^{31} 37^{37} 41^{41}.$

Schinzel's Theorem: If there is an $f(x) \in \mathbb{Z}[x]$ such that $f(1) \neq -1$ and $f(x)x^n + 1$ is reducible for all non-negative integers n, then there is an odd covering of the integers.

TURÁN'S CONJECTURE

Conjecture: There is an absolute constant *C* such that if

$$f(x)=\sum_{j=0}^r a_j x^j\in \mathbb{Z}[x],$$

then there is a

$$g(x) = \sum_{j=0}^r b_j x^j \in \mathbb{Z}[x]$$

irreducible (over \mathbb{Q}) such that $\sum_{j=0}^{r} |b_j - a_j| \leq C$.

Conjecture: There is an absolute constant *C* such that if

$$f(x)=\sum_{j=0}^r a_j x^j\in \mathbb{Z}[x],$$

then there is a

$$g(x) = \sum_{j=0}^r b_j x^j \in \mathbb{Z}[x]$$

irreducible (over \mathbb{Q}) such that $\sum_{j=0}^{\cdot} |b_j - a_j| \leq C$.

Comment: The conjecture remains open. If we take $g(x) = \sum_{j=0}^{s} b_j x^j \in \mathbb{Z}[x]$ where possibly s > r, then the problem has been resolved by Schinzel.

Old Theorem: When m is large, either $u(x)x^m + v(x)$ has an obvious factorization or the non-reciprocal part of $u(x)x^m + v(x)$ is irreducible.

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If one can show g(x) is irreducible for some n, then the conjecture of Turán (modified so $\deg g > \deg f$ is allowed) is resolved with C = 1.

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If f(0) = 0 or f(1) = -1, then consider instead

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Problem: Dealing with $g(x) = x^n + f(x)$ is essentially equivalent to the odd covering problem. So this is hard.

Second Attack on Turán's Problem:

Idea: Consider

$$g(x) = x^m \pm x^n + f(x).$$

If f(0) = 0, then consider instead

$$g(x) = x^m \pm x^n + f(x) \pm 1.$$

Theorem (Schinzel): For every

$$f(x) = \sum_{j=0}^r a_j x^j \in \mathbb{Z}[x],$$

there exist infinitely many irreducible

$$g(x) = \sum_{j=0}^s b_j x^j \in \mathbb{Z}[x]$$

such that

$$\sum_{j=0}^{\max\{r,s\}} |a_j - b_j| \leq \begin{cases} 2 & \text{if } f(0) \neq 0 \\ 3 & \text{always.} \end{cases}$$

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One of these is such that

$$s < \exp \left((5r+7)(\|f\|^2+3)
ight).$$

• Consider $F(x) = x^m + x^n + f(x)$ with $m \in (M, 2M]$ and $n \in (N, 2N]$ where M and N are large and M > N.

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- ► Apply result on u(x)x^m + v(x) with u(x) = 1 and v(x) = xⁿ + f(x) to reduce problem to consideration of reciprocal factors.
- ► Find a bound on the number of x^m + xⁿ + f(x) with reciprocal non-cyclotomic factors.

To bound the x^m + xⁿ + f(x) with cyclotomic factors, set

 $\mathcal{A} = \{(m, n) : M < m \leq 2M, N < n \leq 2N\},\$ and let $\mathcal{A}_p \subset A$ (arising from when $F(\zeta_{p^k}) = 0$). Use a "sieve" argument to estimate the size of

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► To bound the $x^m + x^n + f(x)$ with cyclotomic factors, set

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$$\mathcal{A} - \bigcup \mathcal{A}_p.$$

• Deduce that some $F(x) = x^m + x^n + f(x)$ with $m \in (M, 2M]$ and $n \in (N, 2N]$ is irreducible (where M and N are large and M > N).

Current Knowledge:

Theorem: Given
$$f(x) = \sum_{j=0}^r a_j x^j \in \mathbb{Z}[x]$$
, there are infinitely many irreducible $g(x) = \sum_{j=0}^s b_j x^j \in \mathbb{Z}[x]$

such that max

$$\sum_{j=0}^{lpha\{r,s\}} |a_j - b_j| \leq 5.$$

j=0

One of these is such that

$$s \leq 4r \exp \left(4\|f\|^2 + 12\right).$$

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Theorem: Given
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infinitely many irreducible $g(x) = \sum_{j=0} b_j x^j \in \mathbb{Z}[x]$

such that $\max\{r,s\}$ $\sum_{j=0}^{j=0} |a_j - b_j| \le 5.$

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such that $\max\{r,s\}$
 $\sum_{j=0}^{r} |a_j - b_j| \leq 3.$
One of these is such that

 $s \leq$ some polynomial in r.