3 The Irrationality of $\log 2$

Next, we show that $\log 2$ and $\zeta(3)$ are irrational. The main purpose of proving that $\log 2$ is irrational here is that the proof given is similar to the proof we will give for the irrationality of $\zeta(3)$. In particular, it will be convenient for both arguments to have the following lemma.

Lemma 1. *Let $\epsilon > 0$. Then there is an $N = N(\epsilon)$ such that if $n \geq N$, then*

$$d_n = \text{lcm}\,(1, 2, 3, \ldots, n) < e^{(1+\epsilon)n}.$$

Proof. Observe that if p^r divides a number in $\{1, 2, \ldots, n\}$, then $p^r \leq n$ so that $r \leq \log n / \log p$. On the other hand, $p^{[\log n/\log p]}$ does divide one such number (namely itself). Thus,

$$d_n = \prod_{p \leq n} p^{[\log n/\log p]}.$$

Explain how the rest follows from the Prime Number Theorem. □

Theorem 9. *The number $\log 2$ is irrational.*

Proof. Assume $\log 2 = a/b$ for some integers a and b with $b > 0$. We will obtain a contradiction by showing there are integers c and d for which $0 < |c + d \log 2| < 1/b$. We use that, from Lemma 1, $d_n < 3^n$ for n sufficiently large.

Define

$$I_n = \int_0^1 \frac{x^n(1-x)^n}{(1+x)^{n+1}}\, dx.$$

Observe that the numerator of the integrand is

$$x^n(1-x)^n = \big((1+x)-1\big)^n \big(2-(1+x)\big)^n = \sum_{j=0}^{2n} a_j (1+x)^j,$$

for some integers a_j. When the product of the $2n$ factors $\big((1+x)-1\big)^n\big(2-(1+x)\big)^n$ is expanded to get the last sum above, $(1+x)^j$ occurs when we multiply $\pm(1+x)$ from precisely j of the $2n$ factors together with the constants -1 and 2 from the remaining $2n-j$ factors. In particular, if $j \leq n$, then each such multiplication includes a product of at least $n-j$ of the 2's appearing in this product. In other words, for $j \leq n$, we have $a_j = 2^{n-j}a'_j$ for some integer a'_j.

For $j < n$, we have

$$\int_0^1 a_j(1+x)^{j-n-1}\, dx = \frac{2^{n-j}a'_j(1+x)^{j-n}}{j-n}\bigg|_0^1 = \frac{b_j}{j-n},$$

for some integer b_j. Also, $\int_0^1 a_j(1+x)^{j-n-1}\, dx = b_j/(j-n)$, for $j > n$ and some integers b_j, and $\int_0^1 a_n(1+x)^{-1}\, dx = a_n \log 2$. We deduce that there are integers u_n and v_n such that

$$I_n = \int_0^1 \sum_{j=0}^{2n} a_j(1+x)^{j-n-1}\, dx = \frac{u_n + v_n \log 2}{d_n}.$$

One checks that the maximum of $x(1-x)$ on $[0,1]$ is $1/4$. It follows that

$$0 < |u_n + v_n \log 2| = |I_n d_n| < (1/4)^n d_n < (1/4)^n 3^n < (3/4)^n.$$

Taking n so that $(3/4)^n < 1/b$, $c = u_n$, and $d = v_n$, we obtain the contradiction we sought. □