9 Weyl's Theorem

We will give some preliminary material before introducing Weyl's Theorem and its proof. In particular, the following result is fairly straightforward, but nevertheless it leads to some nice examples.

Theorem 23. *Let k be an integer. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}\,dx = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases}$$

Equivalently,

$$\int_0^{1} e^{i2\pi kx}\,dx = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases}$$

Also, if $n > |k|$, then

$$\sum_{j=0}^{n-1} e^{i2\pi kj/n} = \begin{cases} 0 & \text{if } k \neq 0 \\ n & \text{if } k = 0. \end{cases}$$

We give a few examples of the usefulness of Theorem 23. We do this by posing problems and demonstrating a solution to each based on the above theorem. It should be kept in mind that we do not mean to imply the the following solutions are the most elegant.

Example 1 (Putnam A-5, 1985): Let

$$I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cos(3x)\cdots\cos(mx)\,dx.$$

For which integers m, $1 \leq m \leq 10$, is $I_m \neq 0$?

Solution: Recall that

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}.$$

Hence,

$$I_m = \int_0^{2\pi}\prod_{k=1}^{m}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{1}{2^m}\sum_{\epsilon_1,\ldots,\epsilon_m\in\{1,-1\}}\int_0^{2\pi} e^{i(\epsilon_1+2\epsilon_2+\cdots+m\epsilon_m)x}\,dx.$$

From Theorem 23, we deduce that each of the integrals in the sum is 0 unless $\epsilon_1+2\epsilon_2+\cdots+m\epsilon_m = 0$. Also, if an integral in the sum is not zero, then it is positive. Therefore, $I_m \neq 0$ if and only if there exist $\epsilon_1,\ldots,\epsilon_m \in \{1,-1\}$ such that $\epsilon_1+2\epsilon_2+\cdots+m\epsilon_m = 0$. Note that if $\epsilon_1+2\epsilon_2+\cdots+m\epsilon_m = 0$, then

$$0 \equiv \epsilon_1+2\epsilon_2+\cdots+m\epsilon_m \equiv 1+2+\cdots+m \equiv \frac{m(m+1)}{2} \pmod{2}$$

so that

$$m \equiv 0 \text{ or } 3 \pmod{4}.$$

Thus, $I_m = 0$ for $m \in \{1, 2, 5, 6, 9, 10\}$. To see that the answer is the remaining m, that is $m \in \{3, 4, 7, 8\}$, observe that

$$1 + 2 - 3 = 0 \quad \text{and } 1 - 2 - 3 + 4 = 0$$

and if

$$\epsilon_1 + 2\epsilon_2 + \cdots + m\epsilon_m = 0,$$

then

$$\epsilon_1 + 2\epsilon_2 + \cdots + m\epsilon_m - (m+1) + (m+2) + (m+3) - (m+4) = 0.$$

Note that in general $I_m \neq 0$ if and only if $m \equiv 0$ or $3 \pmod 4$.

Example 2 (Putnam A-6, 1985): If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + \cdots + a_m^2.$$

Let $f(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$, and

(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$ for all positive integers n.

Solution: Observe that Theorem 23 implies

$$\begin{aligned}\Gamma(p(x)) &= a_0^2 + \cdots + a_m^2 \\ &= \int_0^1 \left(a_0 + a_1 e^{i2\pi x} + \cdots + a_m e^{i2\pi(mx)}\right)\left(a_0 + a_1 e^{-i2\pi x} + \cdots + a_m e^{-i2\pi(mx)}\right) dx \\ &= \int_0^1 p(e^{i2\pi x})\overline{p(e^{i2\pi x})}\, dx.\end{aligned}$$

Thus,

$$\begin{aligned}\Gamma(f(x)^n) &= \Gamma((3x+1)^n(x+2)^n) \\ &= \int_0^1 (3e^{i2\pi x}+1)^n(e^{i2\pi x}+2)^n(3e^{-i2\pi x}+1)^n(e^{-i2\pi x}+2)^n\, dx \\ &= \int_0^1 (3e^{i2\pi x}+1)^n(e^{i2\pi x}+2)^n(e^{-i2\pi x})^n(3e^{-i2\pi x}+1)^n(e^{-i2\pi x}+2)^n(e^{i2\pi x})^n\, dx \\ &= \int_0^1 g(e^{i2\pi x})^n g(e^{-i2\pi x})^n\, dx = \Gamma(g(x)^n),\end{aligned}$$

where $g(x) = (3x+1)(2x+1) = 6x^2 + 5x + 1$. Since $g(0) = 1$, this completes the solution.

Example 3: Let R be a rectangle which is partitioned as a disjoint union (excluding common edges) of rectangles $R_1, \ldots, R_n$ each having sides parallel to the sides of R. Prove that if each R_j has at least one side of integer length for $j = 1, 2, \ldots, n$, then so does R.

Solution: We do not use Theorem 23 directly here but rather the following which is of a similar flavor:

Suppose α and u are real numbers. Then α is an integer if and only if

$$\left| \int_u^{u+\alpha} e^{i2\pi x}\, dx \right| = 0.$$

Position the rectangles so that their sides are parallel to the x and y-axes and so that the lower left corner of R is $(0,0)$. Suppose α_j and β_j are the horizontal and vertical dimensions of R_j, respectively, for $j = 1, 2, \ldots, n$. Let (u_j, v_j) be the lower left corner of R_j. Then since either α_j is an integer or β_j is an integer, we get from the above that

$$\left| \int_{v_j}^{v_j+\beta_j} \int_{u_j}^{u_j+\alpha_j} e^{i2\pi(x+y)}\, dx\, dy \right| = \left| \int_{u_j}^{u_j+\alpha_j} e^{i2\pi x}\, dx \right| \left| \int_{v_j}^{v_j+\beta_j} e^{i2\pi y}\, dy \right| = 0.$$

On the other hand, if α and β are the horizontal and vertical dimensions of R, respectively, then

$$\left| \int_0^{\alpha} e^{i2\pi x}\, dx \right| \left| \int_0^{\beta} e^{i2\pi y}\, dy \right| = \left| \iint_R e^{i2\pi(x+y)}\, dx \right|$$
$$= \left| \sum_{j=1}^{n} \iint_{R_j} e^{i2\pi(x+y)}\, dx \right| = \left| \sum_{j=1}^{n} \int_{v_j}^{v_j+\beta_j} \int_{u_j}^{u_j+\alpha_j} e^{i2\pi(x+y)}\, dx\, dy \right| = 0.$$

Hence, either $\left| \int_0^{\alpha} e^{i2\pi x}\, dx \right| = 0$ or $\left| \int_0^{\beta} e^{i2\pi y}\, dy \right| = 0$, which implies that either α or β is an integer, completing the proof.

We turn now to the main topic of this section. For α real, let $\{\alpha\}$ denote the fractional part of α. Note that $\{\alpha\} \in [0,1)$ for all α. Thus, for example, $\{2.341\} = 0.341$, $\{22/7\} = 1/7$, and $\{-22/7\} = 6/7$. Let $\{\alpha_1, \alpha_2, \ldots\}$ be a sequence of real numbers. We say that the sequence is uniformly distributed modulo one if for every a and b with $0 \le a \le b \le 1$,

$$\lim_{n\to\infty} \frac{|\{r \le n : \{\alpha_r\} \in [a,b]\}|}{n} = b - a.$$

The following result is due to Weyl.

Theorem 24. *If α is a real irrational number, then the sequence $\{r\alpha\}_{r=1}^{\infty}$ is uniformly distributed modulo one.*

Before proving Weyl's Theorem, we discuss some preliminaries. First, we observe that determining whether a sequence is uniformly distributed "may" not even be intuitively clear. For example, suppose $\alpha \in \mathbb{R}$ is irrational. Does it follow that $\{\alpha^r\}_{r=1}^{\infty}$ is uniformly distributed modulo one? One can see that this is not the case by considering $\alpha = \sqrt{2}$ since

$$|\{r \le n : \{\sqrt{2}^r\} \in [0, 1/3]\}| \ge \frac{n-1}{2}.$$

Is this an unfair example since $\alpha = \sqrt{2}$ has the property that some power of α is an integer? To partially answer this question, we show that for $\alpha = (1+\sqrt{5})/2$, the sequence $\{\alpha^r\}_{r=1}^{\infty}$ is not uniformly distributed modulo one. Define

$$u_r = \left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{1-\sqrt{5}}{2}\right)^r \qquad \text{for } r \in \mathbb{Z}^+ \cup \{0\}.$$

Note that $u_0 = 2$, $u_1 = 1$, and for $r \geq 2$,

$$\begin{aligned}
u_r &= \left(\frac{1+\sqrt{5}}{2}\right)^r + \left(\frac{1-\sqrt{5}}{2}\right)^r \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{r-2}\left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1-\sqrt{5}}{2}\right)^{r-2}\left(\frac{1-\sqrt{5}}{2}\right)^2 \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{r-2}\left(\frac{3+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^{r-2}\left(\frac{3-\sqrt{5}}{2}\right) \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{r-2}\left(1+\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right)^{r-2}\left(1+\frac{1-\sqrt{5}}{2}\right) \\
&= u_{r-2} + u_{r-1}.
\end{aligned}$$

Therefore, $u_r \in \mathbb{Z}^+$ for all $r \geq 0$. Note that $(1-\sqrt{5})/2 \in (-1,0)$ so that $\big((1-\sqrt{5})/2\big)^r$ tends to zero as r tends to infinity. Therefore,

$$\lim_{r\to\infty} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^{2r+1} \right\} = 0$$

and

$$\lim_{r\to\infty} \left\{ \left(\frac{1+\sqrt{5}}{2}\right)^{2r} \right\} = 1.$$

This easily implies that

$$\lim_{n\to\infty} \frac{|\{r \leq n : \{\alpha^r\} \in [1/4, 3/4]\}|}{n} = 0$$

so that $\{\alpha^r\}_{r=1}^{\infty}$ is not uniformly distributed modulo one.

We note that it can be shown that, for almost all real numbers α, the sequence $\{\alpha^r\}_{r=1}^{\infty}$ is uniformly distributed modulo 1. On the other hand, no explicit example of such an α is known.

Our proof of Weyl's Theorem will depend on the use of Cesàro sumability. We say that a series $\sum_{j=0}^{\infty} a_j$ has Cesàro sum S if

$$\lim_{n\to\infty} \frac{\sum_{r=0}^{n}\sum_{j=0}^{r} a_j}{n+1} = S.$$

We first make an important connection between Cesàro sums and our usual notion of sums. The next result implies that if a series converges to S, then its Cesàro sum is also S; on the other hand, its Cesàro sum may exist even if the series diverges.

Theorem 25. *Let* $\{a_j\}_{j=0}^{\infty}$ *be a sequence of complex numbers, and set* $s_r = \sum_{j=0}^{r} a_j$ *for* $r \in \mathbb{Z}^+ \cup \{0\}$. *Then*

(i) if $\lim_{r\to\infty} s_r = S$, *then* $\lim_{r\to\infty} \frac{\sum_{r=0}^{n} s_r}{n+1} = S$, *and*

(ii) if $a_j = (-1)^{j(j+1)/2}$, *then* $\lim_{r\to\infty} s_r$ *does not exist and* $\lim_{r\to\infty} \frac{\sum_{r=0}^{n} s_r}{n+1} = 0$.

Proof. (i) Suppose $\lim_{r\to\infty} s_r = S$. Let $\epsilon > 0$. Then there is an R such that if $r \geq R$, then $|s_r - S| < \epsilon/2$. Let $A = \sum_{r=0}^{R} |s_r - S|$. Let $N \geq \max\{R, 2A/\epsilon\}$. Then for $n \geq N$,

$$\begin{aligned}
\left|\frac{\sum_{r=0}^{n} s_r}{n+1} - S\right| &= \frac{1}{n+1}\left|\sum_{r=0}^{n}(s_r - S)\right| \\
&= \frac{1}{n+1}\left|\sum_{r=0}^{R}(s_r - S)\right| + \frac{1}{n+1}\left|\sum_{r=R+1}^{n}(s_r - S)\right| \\
&\leq \frac{1}{n+1}\left(A + (n+1)\frac{\epsilon}{2}\right) \leq \frac{1}{n+1}\left(n\frac{\epsilon}{2} + (n+1)\frac{\epsilon}{2}\right) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus,

$$\lim_{r\to\infty} \frac{\sum_{r=0}^{n} s_r}{n+1} = S.$$

(ii) Here, $\{a_j\}_{j=0}^{\infty} = \{1, -1, -1, 1, 1, -1, -1, 1, \dots\}$ so that $s_0 = 1$, $s_1 = 0$, $s_2 = -1$, $s_3 = 0$, $s_4 = 1, \dots$. Hence, for every $r \geq 0$,

$$|\sum_{r=0}^{n} s_r| \leq 1.$$

Thus,

$$\lim_{r\to\infty} \frac{\sum_{r=0}^{n} s_r}{n+1} = 0.$$

Clearly, $\lim_{r\to\infty} s_r$ does not exist. $\square$

Before continuing, we consider some examples. First, we show that the Cesàro sum of $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is infinite. Let $s_r = \sum_{j=0}^{r}(1/(j+1))$. Note that

$$s_r \geq \int_1^{r+2} \frac{1}{t}\, dt = \log(r+2).$$

Thus,

$$\begin{aligned}
\sum_{r=0}^{n}\sum_{j=0}^{r} a_j = \sum_{r=0}^{n} s_r &\geq \sum_{r=0}^{n} \log(r+2) \\
&\geq \int_1^{n+2} \log t\, dt = t\log t\Big|_1^{n+2} = (n+2)\log(n+2).
\end{aligned}$$

Hence,

$$\frac{\sum_{r=0}^{n}\sum_{j=0}^{r} a_j}{n+1} \geq \frac{(n+2)\log(n+2)}{n+1}.$$

Since this last expression tends to infinity with n, we deduce that the Cesàro sum of $1+\frac{1}{2}+\frac{1}{3}+\cdots$ is infinite.

As another example, we show that it is possible to have a series $\sum_{j=0}^{\infty} a_j$ with all of its partial sums $s_r = \sum_{j=0}^{r} a_j$ bounded and with no Cesàro sum. We take

$$a_j = \begin{cases} 1 & \text{if } j = 2^{2k} \text{ where } k \in \mathbb{Z}^+ \cup \{0\} \\ -1 & \text{if } j = 2^{2k-1} \text{ where } k \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases}.$$

Then we get that $a_0, a_1, a_2, \dots$ is $0, 1, -1, 0, 1, 0, 0, 0, -1, \dots$, and $s_0, s_1, s_2, \dots$ is $0, 1, 0, 0, 1, 1, 1, 1, 0, \dots$. In general,

$$s_j = \begin{cases} 1 & \text{if } j \in [2^{2k}, 2^{2k+1}) \cap \mathbb{Z} \text{ where } k \in \mathbb{Z}^+ \cup \{0\} \\ 0 & \text{otherwise} \end{cases}.$$

Note that for any positive integer m, $1 + 2 + 2^2 + \cdots + 2^m < 2^{m+1}$ so that

$$\sum_{j=0}^{2^{2k}-1} s_j = \sum_{j=0}^{k-1} 2^{2j} \leq \left(\sum_{j=0}^{2k-4} 2^j\right) + 2^{2k-2}$$

$$\leq 2^{2k-3} + 2^{2k-2} = \frac{3}{8}(2^{2k}).$$

Also,

$$\sum_{j=0}^{2^{2k+1}-1} s_j = \sum_{j=0}^{k} 2^{2j} \geq 2^{2k} = \frac{1}{2}(2^{2k+1}).$$

Thus, infinitely often $\left(\sum_{j=0}^{n} a_j\right)/(n+1) \leq 3/8$ and infinitely often $\left(\sum_{j=0}^{n} a_j\right)/(n+1) \geq 1/2$. This easily implies that the Cesàro sum cannot exist.

To prove Weyl's Theorem (Theorem 24), we will use a little material from Fourier Analysis which we introduce here. The basic idea is to write a function $f(x)$, which maps the real numbers to the complex numbers, in the form

$$f(x) = \sum_{r=-\infty}^{\infty} \hat{f}(r) e^{irx}. \tag{11}$$

Here, we wish to find numbers $\hat{f}(r)$ for which the above holds. Since $e^{irx} = \cos(rx) + i\sin(rx)$, the series on the right in (11) is often referred to as a trignometric series. It is not always possible to obtain a trignometric series representation for $f(x)$, but let's suppose for the moment that $f(x)$ can be expressed in the form (11). We temporarily ignore rigor. Using Theorem 23, observe that for $n \geq |r|$,

$$\hat{f}(r) = \frac{1}{2\pi} \sum_{j=-n}^{n} \int_0^{2\pi} \hat{f}(j) e^{i(j-r)x}\, dx = \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{j=-n}^{n} \hat{f}(j) e^{ijx}\right) e^{-irx}\, dx.$$

Letting n tend to infinity, we deduce that

$$\hat{f}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-irx}\, dx. \tag{12}$$

Thus, using (12), it is seemingly easy to write a function in the form (11). However, we still need to discuss when the series in (11) converges. We will require certian conditions on $f(x)$. They are:

(i) $f(x)$ is continuous over the reals.

(ii) $f(x + 2\pi) = f(x)$ for all real x.

The latter condition shouldn't be surprising since the values of $\hat{f}(r)$ determine $f(x)$ by (11) and they only depend on the values of $f(x)$ for $x \in [0, 2\pi]$ by (12). Before continuing, it is worth noting that (i) and (ii) are not sufficient to imply the convergence of the series in (11). On the other hand, we will want to consider $f(x)$ in this generality. To deal with this difficulty, we prove a result of Fejér that with $f(x)$ satisfying (i) and (ii), the series in (11) has Cesàro sum $f(x)$ (so that convergence of the series will not be necessary).

Define $\hat{f}(r)$ by (12), and set

$$\begin{aligned}
\sigma_n(f, x) &= \frac{1}{n+1} \sum_{j=0}^{n} \left(\sum_{r=-j}^{j} \hat{f}(r) e^{irx} \right) \\
&= \frac{1}{n+1} \Big((n+1)\hat{f}(0) + n\hat{f}(1)e^{ix} + n\hat{f}(-1)e^{-ix} + \cdots \Big) \\
&= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \hat{f}(r) e^{irx}.
\end{aligned}$$

We will now prove

Theorem 26. *If $f(x)$ satisfies (i) and (ii), then $\sigma_n(f, x)$ converges to f uniformly on $\mathbb{R}$.*

We begin with a relationship between $\sigma_n(f, x)$ and

$$K_n(x) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{irx}.$$

Observe that

$$\begin{aligned}
\sigma_n(f, x) &= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \hat{f}(r) e^{irx} \\
&= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} \frac{1}{2\pi} \left(\int_0^{2\pi} f(t) e^{-irt}\, dt \right) e^{irx} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{ir(x-t)}\, dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(t) K_n(x-t)\, dt.
\end{aligned}$$

Note that in addition to (ii), we have that $K_n(x+2\pi) = K_n(x)$ for all real x. Letting $y = x - t$ gives

$$\begin{aligned}\sigma_n(f,x) &= \frac{-1}{2\pi}\int_x^{x-2\pi} f(x-y)K_n(y)\,dy \\ &= \frac{1}{2\pi}\int_{x-2\pi}^{x} f(x-y)K_n(y)\,dy = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)K_n(y)\,dy.\end{aligned}$$

We now proceed with two lemmas.

Lemma 1. *Let* $x \in [0, 2\pi)$*. Then* $K_n(0) = n+1$*, and if* $x \neq 0$*, then*

$$K_n(x) = \frac{1}{n+1}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2.$$

Proof. Clearly,

$$K_n(0) = \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} = \frac{1}{n+1}\left(n+1+2\sum_{r=1}^{n} r\right) = n+1.$$

Now, suppose $x \in (0, 2\pi)$. Then

$$\begin{aligned}K_n(x) &= \sum_{r=-n}^{n} \frac{n+1-|r|}{n+1} e^{irx} \\ &= \frac{1}{n+1}\left((n+1)e^{-inx}\sum_{r=-n}^{n} e^{i(r+n)x} - \sum_{r=1}^{n}\left(re^{irx} + re^{-irx}\right)\right).\end{aligned}$$

Note that

$$\sum_{r=-n}^{n} e^{i(r+n)x} = \sum_{r=0}^{2n} e^{irx} = \frac{e^{i(2n+1)x}-1}{e^{ix}-1}.$$

Also,

$$\sum_{r=1}^{n} e^{irx} = \frac{e^{i(n+1)x}-1}{e^{ix}-1} - 1$$

so that by taking derivatives, we obtain

$$i\sum_{r=1}^{n} re^{irx} = \frac{(e^{ix}-1)i(n+1)e^{i(n+1)x} - (e^{i(n+1)x}-1)ie^{ix}}{(e^{ix}-1)^2}.$$

Thus,

$$\sum_{r=1}^{n} re^{irx} = \frac{ne^{i(n+2)x} - (n+1)e^{i(n+1)x} + e^{ix}}{(e^{ix}-1)^2}$$

and

$$\sum_{r=1}^{n} re^{-irx} = \frac{ne^{-i(n+2)x} - (n+1)e^{-i(n+1)x} + e^{-ix}}{(e^{-ix}-1)^2}.$$

Hence,

$$\begin{aligned}
K_n(x) &= \frac{1}{(n+1)(e^{ix}-1)^2}\big((n+1)e^{-inx}(e^{ix}-1)(e^{i(2n+1)x}-1) \\
&\qquad - \big(ne^{i(n+2)x} - (n+1)e^{i(n+1)x} + e^{ix}\big) - \big(ne^{-inx} - (n+1)e^{-i(n-1)x} + e^{ix}\big)\big) \\
&= \frac{1}{(n+1)(e^{ix}-1)^2}\big(e^{i(n+2)x} + e^{-inx} - 2e^{ix}\big) \\
&= \frac{e^{ix}}{(n+1)(e^{ix}-1)^2}\big(e^{i(n+1)x} - 2 + e^{-i(n+1)x}\big) \\
&= \frac{e^{ix}}{(n+1)(e^{ix}-1)^2}\big(e^{i(n+1)x/2} - e^{-i(n+1)x/2}\big)^2 \\
&= \frac{1}{n+1}(e^{ix/2} - e^{-ix/2})^{-2}\big(e^{i(n+1)x/2} - e^{-i(n+1)x/2}\big)^2 \\
&= \frac{1}{n+1}\left(\frac{e^{ix/2} - e^{-ix/2}}{2i}\right)^{-2}\left(\frac{e^{i(n+1)x/2} - e^{-i(n+1)x/2}}{2i}\right)^2 \\
&= \frac{1}{n+1}\left(\frac{\sin((n+1)x/2)}{\sin(x/2)}\right)^2.
\end{aligned}$$

This completes the proof. □

Lemma 2. *$K_n(x)$ has the following properties:*

(a) $K_n(x) \geq 0$ *for all real numbers* x.

(b) *For every* $\delta > 0$ *and* $\epsilon > 0$, *there is an* $N = N(\delta, \epsilon)$ *such that if* $n \geq N$ *and* $x \in (\delta, 2\pi - \delta)$, *then* $|K_n(x)| < \epsilon$.

(c) $\frac{1}{2\pi}\int_0^{2\pi} K_n(x)\,dx = 1$.

Comment: (b) is simply asserting that for all $\delta > 0$, $K_n(x)$ approaches 0 uniformly on $(\delta, 2\pi - \delta)$.

Proof. Since $K_n(x+2\pi) = K_n(x)$, (a) follows from Lemma 1. For (b), note that for $x \in (\delta, 2\pi-\delta)$ and for n sufficiently large (independent of x), Lemma 1 implies

$$K_n(x) \leq \frac{1}{(n+1)\sin^2(x/2)} \leq \frac{1}{(n+1)\sin^2(\delta/2)} < \epsilon.$$

Finally, we deduce (c) from

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} K_n(x)\,dx &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{r=-n}^{n}\frac{n+1-|r|}{n+1}e^{irx}\,dx \\
&= \sum_{r=-n}^{n}\frac{n+1-|r|}{n+1}\frac{1}{2\pi}\int_0^{2\pi} e^{irx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} dx = 1,
\end{aligned}$$

completing the proof. □

We now give the basic idea behind the proof of Theorem 26. Note that Lemma 2 implies that for $\delta > 0$ fixed and small and for n large,

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} K_n(y)\,dy \approx 0 \quad \text{and} \quad \frac{1}{2\pi}\int_{0}^{2\pi} K_n(y)\,dy = 1$$

so that

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(y)\,dy \approx 1.$$

Also,

$$\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} f(x-y)K_n(y)\,dy \approx 0$$

so that

$$\sigma_n(f,x) = \frac{1}{2\pi}\int_{0}^{2\pi} f(x-y)K_n(y)\,dy \approx \frac{1}{2\pi}\int_{-\delta}^{\delta} f(x-y)K_n(y)\,dy.$$

Since $f(x)$ is continuous, $f(x-y) \approx f(x)$ for $y \in [-\delta,\delta]$. Thus,

$$\sigma_n(f,x) \approx \frac{1}{2\pi}\int_{-\delta}^{\delta} f(x)K_n(y)\,dy = f(x)\left(\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(y)\,dy\right) \approx f(x).$$

We now make the above ideas rigorous.

Proof of Theorem 26. We use that $f(x)$ is bounded and uniformly continuous on the compact interval $[0,4\pi]$. Thus, there is an M such that $|f(x)| \le M$ for all $x \in [0,4\pi]$; and for all $\epsilon > 0$, there is a $\delta > 0$ such that if x and y are in $[0,4\pi]$ with $|x-y| < \delta$, then $|f(x)-f(y)| < \epsilon$. By property (ii), $|f(x)| \le M$ for all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x)-f(y)| < \epsilon$ for all x and y in $\mathbb{R}$ with $|x-y| < \delta$.

Let $\epsilon > 0$. Let $\delta \in (0,\pi)$ such that $|f(x)-f(y)| < \epsilon/2$ for all x and y in $\mathbb{R}$ with $|x-y| < \delta$. By Lemma 2 (b), there is an $N = N(\delta,\epsilon)$ such that if $n \ge N$ and $x \in (\delta, 2\pi-\delta)$, then $|K_n(x)| \le \epsilon/(4M)$. By Lemma 2 (a),

$$\int_{-\delta}^{\delta} |K_n(y)|\,dy = \int_{-\delta}^{\delta} K_n(y)\,dy \le \int_{0}^{2\pi} |K_n(y)|\,dy.$$

Thus, we get from Lemma 2 (c) that for all $n \geq N$,

$$
\begin{aligned}
|\sigma_n(f,x) - f(x)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(x-y) K_n(y)\, dy - \left(\frac{1}{2\pi} \int_0^{2\pi} K_n(y)\, dy \right) f(x) \right| \\
&= \left| \frac{1}{2\pi} \int_0^{2\pi} \left(f(x-y) - f(x) \right) K_n(y)\, dy \right| \\
&\leq \frac{1}{2\pi} \left| \int_{-\delta}^{\delta} \left(f(x-y) - f(x) \right) K_n(y)\, dy \right| + \frac{1}{2\pi} \left| \int_{\delta}^{2\pi-\delta} \left(f(x-y) - f(x) \right) K_n(y)\, dy \right| \\
&\leq \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-y) - f(x)|\, |K_n(y)|\; dy + \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} \left(|f(x-y)| + |f(x)| \right) |K_n(y)|\; dy \\
&< \frac{\epsilon}{2} \frac{1}{2\pi} \int_{-\delta}^{\delta} |K_n(y)|\; dy + (2M) \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} \frac{\epsilon}{4M}\, dy \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \left(\frac{1}{2\pi} \int_0^{2\pi} dy \right) = \epsilon.
\end{aligned}
$$

The above inequality is independent of x, establishing the Theorem. □

Corollary 1. *Let $f(x)$ satisfy (i) and (ii), and let $\epsilon > 0$. Then there exists a trignometric polynomial*

$$P(x) = \sum_{j=-n}^{n} a_j e^{ijx}$$

(with $a_j \in \mathbb{C}$ for each j) such that

$$\sup_{x \in \mathbb{R}} |P(x) - f(x)| \leq \epsilon.$$

Proof. Since $\sigma_n(f,x)$ is a trignometric polynomial, the Corollary follows from Theorem 26 by taking $P(x) = \sigma_n(f,x)$ with n sufficiently large. □

We now prove Weyl's Theorem (Theorem 24). It suffices to show that if $\alpha \in \mathbb{R}$ is irrational and $0 \leq a \leq b \leq 1$, then

$$\lim_{n \to \infty} \frac{|\{r \leq n : 2\pi\{r\alpha\} \in [2\pi a, 2\pi b]\}|}{n} = b - a. \tag{13}$$

Lemma 3. *Let $\alpha \in \mathbb{R}$ with α irrational. Suppose $f : \mathbb{R} \to \mathbb{C}$ satisfies (i) and (ii) of the previous section. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} f(2\pi r\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx.$$

Proof. Let

$$G_n(f) = \frac{1}{n} \sum_{r=1}^{n} f(2\pi r\alpha) - \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx.$$

We want to show that $\lim_{n\to\infty} G_n(f) = 0$. First, we consider $f(x) = e^{isx}$ where s is an integer. If $s = 0$, then $f(x) \equiv 1$ and

$$G_n(f) = G_n(1) = \frac{1}{n}\sum_{r=1}^{n} 1 - \frac{1}{2\pi}\int_0^{2\pi} dx = 0.$$

Thus, $\lim_{n\to\infty} G_n(f) = 0$ in the case $f(x) = e^{isx}$ with $s = 0$. Now, suppose $s \neq 0$. Then

$$\begin{aligned}
|G_n(f)| &= \left|\frac{1}{n}\sum_{r=1}^{n} e^{2\pi irs\alpha} - \frac{1}{2\pi}\int_0^{2\pi} e^{isx}\,dx\right| \\
&= \left|\frac{1}{n}\sum_{r=1}^{n} e^{2\pi irs\alpha}\right| \\
&= \frac{1}{n}\left|e^{2\pi is\alpha}\right|\left|\frac{e^{2\pi ins\alpha}-1}{e^{2\pi is\alpha}-1}\right| \\
&\leq \frac{2}{n|e^{2\pi is\alpha}-1|}.
\end{aligned}$$

Thus, in this case, $\lim_{n\to\infty} G_n(f) = 0$.

Now, we consider the case when $f(x) = \sum_{s=-m}^{m} a_s e^{isx}$ (i.e., $f(x)$ is a trignometric polynomial). Then

$$\begin{aligned}
\lim_{n\to\infty} G_n(f) &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{n}\sum_{s=-m}^{m} a_s e^{2\pi irs\alpha} - \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{s=-m}^{m} a_s e^{isx}\right)dx\right) \\
&= \sum_{s=-m}^{m} a_s \lim_{n\to\infty}\left(\frac{1}{n}\sum_{r=1}^{n} e^{2\pi irs\alpha} - \frac{1}{2\pi}\int_0^{2\pi} e^{isx}\,dx\right) \\
&= \sum_{s=-m}^{m} a_s \lim_{n\to\infty} G_n\left(e^{isx}\right) = 0.
\end{aligned}$$

Thus, the lemma holds for trignometric polynomials.

We now consider the general case when $f(x)$ satisfies (i) and (ii). Let $\epsilon > 0$. We show that if n is sufficiently large, then $|G_n(f)| < \epsilon$. By the Corollary to Theorem 26, there is a trignometric polynomial $P(x)$ such that

$$|f(x) - P(x)| < \frac{\epsilon}{3} \quad \text{for all real } x.$$

Also, since the lemma has already been established for trignometric polynomials, there is an N

such that if $n \geq N$, then $|G_n(P)| < \epsilon/3$. Hence, for all $n \geq N$,

$$\begin{aligned}
|G_n(f)| &\leq |G_n(P)| + |G_n(f) - G_n(P)| \\
&< \frac{\epsilon}{3} + \left| \frac{1}{n} \sum_{r=1}^{n} \left(f(2\pi r\alpha) - P(2\pi r\alpha)\right) - \frac{1}{2\pi} \int_0^{2\pi} \left(f(x) - P(x)\right)\, dx \right| \\
&\leq \frac{\epsilon}{3} + \frac{1}{n} \sum_{r=1}^{n} |f(2\pi r\alpha) - P(2\pi r\alpha)| + \frac{1}{2\pi} \int_0^{2\pi} |f(x) - P(x)|\, dx \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{1}{2\pi} \int_0^{2\pi} \frac{\epsilon}{3}\, dx = \epsilon,
\end{aligned}$$

completing the proof. □

Proof of Theorem 24. Let $\epsilon > 0$. We will apply Lemma 3 to two functions $f_+(x)$ and $f_-(x)$ satisfying (ii). Their precise definitions are not important; a rough graph suffices for the proof. But since graphs are more difficult to print than precise definitions in TEX, we leave it to the reader to graph

$$f_+(x) = \begin{cases} 0 & \text{if } x \in [0, 2\pi(a-\epsilon)) \cup [2\pi(b+\epsilon), 2\pi) \\ \dfrac{1}{2\pi\epsilon}x - \dfrac{1}{\epsilon}(a-\epsilon) & \text{if } x \in [2\pi(a-\epsilon), 2\pi a) \\ 1 & \text{if } x \in [2\pi a, 2\pi b) \\ -\dfrac{1}{2\pi\epsilon}x + \dfrac{1}{\epsilon}(b+\epsilon) & \text{if } x \in [2\pi b, 2\pi(b+\epsilon)) \end{cases}$$

and

$$f_+(x) = \begin{cases} 0 & \text{if } x \in [0, 2\pi a) \cup [2\pi b, 2\pi) \\ \dfrac{1}{2\pi\epsilon}x - \dfrac{1}{\epsilon}a & \text{if } x \in [2\pi a, 2\pi(a+\epsilon)) \\ 1 & \text{if } x \in [2\pi(a+\epsilon), 2\pi(b-\epsilon)) \\ -\dfrac{1}{2\pi\epsilon}x + \dfrac{1}{\epsilon}b & \text{if } x \in [2\pi(b-\epsilon), 2\pi b) \end{cases}.$$

Note that $f_+(x)$ and $f_-(x)$ are defined for all real numbers x by the above and (ii). Thus,

$$\sum_{r=1}^{n} f_+(2\pi r\alpha) \geq |\{r \leq n : 2\pi\{r\alpha\} \in [2\pi a, 2\pi b]\}| \geq \sum_{r=1}^{n} f_-(2\pi r\alpha).$$

By Lemma 3, there is an N such that if $n \geq N$, then

$$\frac{1}{n} \sum_{r=1}^{n} f_+(2\pi r\alpha) \leq \frac{1}{2\pi} \int_0^{2\pi} f_+(x)\, dx + \epsilon = \frac{1}{2\pi}\left(2\pi(b-a) + 2\pi\epsilon\right) + \epsilon = (b-a) + 2\epsilon$$

and

$$\frac{1}{n} \sum_{r=1}^{n} f_-(2\pi r\alpha) \geq \frac{1}{2\pi} \int_0^{2\pi} f_-(x)\, dx - \epsilon = \frac{1}{2\pi}\left(2\pi(b-a) - 2\pi\epsilon\right) - \epsilon = (b-a) - 2\epsilon.$$

Hence, for all $n \geq N$,

$$\left| \frac{|\{r \leq n : 2\pi\{r\alpha\} \in [2\pi a, 2\pi b]\}|}{n} - (b-a) \right| < 2\epsilon,$$

from which Theorem 24 follows. $\square$

The following are two problems related to this subject. The details of their solutions are omitted here.

(1) Let $d \in \{0, 1, 2, \ldots, 9\}$. What is the proportion of times that 2^n begins with the digit d as n runs through the positive integers? More specifically, compute

$$\lim_{n\to\infty} \frac{|\{n \leq x : 2^n \text{ begins with the digit } d\}|}{x}.$$

(By the way, of the first 1000 values of 2^n beginning with $n = 0$, exactly 301 begin with the digit 1, and $\log_{10} 2 = 0.3010\ldots$.)

(2) (*Monthly Problem*, 1986) Suppose $x \in \mathbb{R}$ with $x > 1$. Let $a_n = [x^n]$ where n is a positive integer. Let $S = 0.a_1a_2a_3\ldots$. (For example, if $x = \pi$, then $S = 0.393197\ldots$ since $[\pi] = 3$, $[\pi^2] = 9$, $[\pi^3] = 31$, and $[\pi^4] = 97$.) Prove that S is irrational.

Homework:

1. Let k and n be positive integers with $n > 2k$. Prove that

$$\sum_{j=0}^{n-1} \cos^{2k}\left(\frac{2\pi j}{n}\right) = \frac{\binom{2k}{k}}{2^{2k}} n.$$

   (Hint: Use the binomial theorem.)

2. Let $\{\alpha_1, \alpha_2, \ldots\}$ be a sequence of real numbers. Prove that the sequence is uniformly distributed modulo one if and only if for every a and b with $0 \leq a \leq b \leq 1$,

$$\lim_{n\to\infty} \frac{|\{r \leq n : \{\alpha_r\} \in (a,b)\}|}{n} = b - a.$$

3. Let $\sum_{j=0}^{\infty} a_j$ be a divergent series with $a_j > 0$ for each $j \geq 0$. Prove that the Cesàro sum of the series is infinite.

4. Calculate the Cesàro sum of each of the following series.

   (a) $1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

   (b) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

5. Prove the following:

Theorem 27. *Let $\{\alpha_k\}$ be a sequence of real numbers satisfying*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m \alpha_k} = 0$$

for every non-zero integer m. Then $\{\alpha_k\}$ is uniformly distributed modulo 1.

(Hint: Look at the proof of Theorem 24 with α_k replacing $k\alpha$ and decide what changes need to be made. You do not need to rewrite the proofs if you point out clearly where the changes need to be made and what the changes are.)

6. Prove that the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ is uniformly distributed modulo 1.

7. Prove that the sequence $\{\log n\}_{n=1}^{\infty}$ is not uniformly distributed modulo 1.

8. Show that there exists a sequence $\{a_j\}_{j=0}^{\infty}$ such that the partial sums $s_r = \sum_{j=0}^{r} a_j$ satisfy both

$$\liminf_{n\to\infty} \frac{\sum_{r=0}^{n} s_r}{n+1} = -\infty$$

and

$$\limsup_{n\to\infty} \frac{\sum_{r=0}^{n} s_r}{n+1} = +\infty.$$