3 The Irrationality of $\log 2$

Next, we show that $\log 2$ and $\zeta(3)$ are irrational. The main purpose of proving that $\log 2$ is irrational here is that the proof given is similar to the proof we will give for the irrationality of $\zeta(3)$. In particular, it will be convenient for both arguments to have the following lemma.

Lemma 1. *Let $\epsilon > 0$. Then there is an $N = N(\epsilon)$ such that if $n \geq N$, then*

$$d_n = \text{lcm}\,(1, 2, 3, \ldots, n) < e^{(1+\epsilon)n}.$$

Proof. Observe that if p^r divides a number in $\{1, 2, \ldots, n\}$, then $p^r \leq n$ so that $r \leq \log n / \log p$. On the other hand, $p^{[\log n/\log p]}$ does divide one such number (namely itself). Thus,

$$d_n = \prod_{p \leq n} p^{[\log n/\log p]}.$$

Explain how the rest follows from the Prime Number Theorem. □

Theorem 9. *The number $\log 2$ is irrational.*

Proof. Assume $\log 2 = a/b$ for some integers a and b with $b > 0$. We will obtain a contradiction by showing there are integers c and d for which $0 < |c + d \log 2| < 1/b$. We use that, from Lemma 1, $d_n < e^{3n/2}$ for n sufficiently large.

Observe that one of $x^n + 1$ and $x^n - 1$ is divisible by $x + 1$ in $\mathbb{Z}[x]$. Thus,

$$\frac{x^n}{1+x} = f(x) \pm \frac{1}{1+x},$$

for some $f(x) \in \mathbb{Z}[x]$. Upon integration, one obtains

$$\int_0^1 \frac{x^n}{1+x}\,dx = \frac{u_n + v_n \log 2}{d_n} \tag{5}$$

for some integers u_n and v_n. Define

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}\big(x^n(1-x)^n\big)$$

(the n^{th} Legendre polynomial). Note that $P_n(x) \in \mathbb{Z}[x]$. Define

$$I_n = \int_0^1 \frac{x^n(1-x)^n}{(1+x)^{n+1}}\,dx.$$

By repeated integration by parts,

$$I_n = \int_0^1 \frac{P_n(x)}{1+x}\,dx.$$

It follows from (5) that there are integers u_n' and v_n' such that

$$I_n = \frac{u_n' + v_n' \log 2}{d_n}.$$

One checks that the maximum of $x(1-x)/(1+x)$ on $[0, 1]$ is $(\sqrt{2} - 1)^2$. It follows that

$$0 < |u_n' + v_n' \log 2| = |I_n d_n| < (\sqrt{2} - 1)^{2n} d_n < (0.172)^n (4.49)^n < (0.8)^n.$$

Taking n so that $(0.8)^n < 1/b$, $c = u_n'$, and $d = v_n'$, we obtain the contradiction we sought. □