2 The Irrationality of π and Various Trignometric Values

Lemma 1. *Let $f(x) \in \mathbb{Z}[x]$. Then for any non-negative integer n and any integer k, $n!$ divides $f^{(n)}(k)$.*

Proof. Write

$$f(x) = a_t x^t + a_{t-1} x^{t-1} + \cdots + a_1 x + a_0.$$

Then

$$f^{(n)}(k) = \sum_{m=0}^{t} a_m m(m-1)\cdots(m-n+1)k^{m-n} = \sum_{m=0}^{t} a_m n! \binom{m}{n} k^{m-n},$$

from which the result follows. □

Theorem 7. *The number π is irrational.*

Proof. It suffices to prove that π^2 is irrational. Let n be a positive integer which we will choose shortly, and let

$$f(x) = \frac{x^n(1-x)^n}{n!}.$$

We make a couple observations. First,

$$0 < f(x) < \frac{1}{n!} \quad \text{for every } x \in (0,1). \tag{2}$$

Secondly, by expanding $f(x)$ as a polynomial in x and then as a polynomial in $1-x$ (for the latter, replace x with $1-(1-x)$), we deduce with the help of Lemma 1 that

$$f^{(j)}(0) \in \mathbb{Z} \quad \text{and} \quad f^{(j)}(1) \in \mathbb{Z} \quad \text{for every } j \in \mathbb{Z}^+ \cup \{0\}.$$

Assume now that $\pi^2 = a/b$ for some integers a and b with $b > 0$. Let

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x) \right).$$

Then

$$F(0) \in \mathbb{Z} \quad \text{and} \quad F(1) \in \mathbb{Z}.$$

Also,

$$F^{(2)}(x) + \pi^2 F(x) = b^n \pi^{2n+2} f(x) = a^n \pi^2 f(x).$$

Thus,

$$\frac{d}{dx} \left(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x) \right) = \left(F^{(2)}(x) + \pi^2 F(x) \right) \sin(\pi x) = a^n \pi^2 f(x) \sin(\pi x).$$

Hence,

$$\pi a^n \int_0^1 f(x)\sin(\pi x)dx = \frac{1}{\pi} \left(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x) \right)_0^1 = F(1) + F(0).$$

Hence, the expression on the left above is an integer. On the other hand, we get from (2) that

$$0 < \pi a^n \int_0^1 f(x)\sin(\pi x)dx < \frac{\pi a^n}{n!}.$$

We obtain a contradiction by choosing n sufficiently large so that $\pi a^n/n! < 1$. Therefore, π is irrational. □

The next result is more general than the last one but also a little more difficult to prove. The basic idea behind the proof is, however, the same. Theorem 7 follows from Theorem 8 by taking $\alpha = \pi$ in Theorem 8.

Theorem 8. *For any rational number $\alpha \neq 0$, $\cos\alpha$ is irrational.*

Lemma 2. *Let r be any number. Suppose $f(x)$ is a polynomial in $(r-x)^2$, that is $f(x)$ can be written in the form*

$$f(x) = a_{2n}(r-x)^{2n} + a_{2n-2}(r-x)^{2n-2} + \cdots + a_2(r-x)^2 + a_0.$$

Then for any positive odd integer k, $f^{(k)}(r) = 0$.

Proof. Observe that the result is true for each term in the expression for $f(x)$; hence, the lemma follows. □

Proof of Theorem 8. Since $\cos(\alpha) = \cos(-\alpha)$, it suffices to consider $\alpha > 0$. Write $\alpha = a/b$ where a and b are positive integers. Let n be a positive integer to be specified later, and define

$$f(x) = \frac{x^{n-1}(a-bx)^{2n}(2a-bx)^{n-1}}{(n-1)!} = \frac{(\alpha-x)^{2n}\left(\alpha^2-(\alpha-x)^2\right)^{n-1}b^{3n-1}}{(n-1)!}.$$

Then

$$0 < f(x) < \frac{\alpha^{2n}(\alpha^2)^{n-1}b^{3n-1}}{(n-1)!} = \frac{\alpha^{4n-2}b^{3n-1}}{(n-1)!} \quad \text{for } 0 < x < \alpha.$$

Define

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots - f^{(4n-2)}(x).$$

Then

$$F^{(2)}(x) + F(x) = f(x)$$

so that

$$\frac{d}{dx}\left(F'(x)\sin x - F(x)\cos x\right) = F^{(2)}(x)\sin x + F(x)\sin x = f(x)\sin x.$$

Hence,

$$\int_0^\alpha f(x)\sin x\,dx = F'(\alpha)\sin\alpha - F(\alpha)\cos\alpha + F(0). \tag{3}$$

Since $f(x)$ is a polynomial in $(\alpha-x)^2$, we get from Lemma 2 and the definition of $F(x)$ that $F'(\alpha) = 0$. Since $f(x)$ is divisible by x^{n-1}, we deduce from Lemma 1 that $F(0)$ is an integer. Similarly, viewing $f(x)$ as a polynomial in $\alpha - x$, we can get that $F(\alpha)$ is an integer; to clarify any concerns about the denominator of α somehow coming into play, we note that Lemma 1 applies more directly with

$$f(\alpha-x) = \frac{x^{2n}\left(\alpha^2-x^2\right)^{n-1}b^{3n-1}}{(n-1)!} = \frac{x^{2n}\left(a^2-b^2x^2\right)^{n-1}b^{n+1}}{(n-1)!}.$$

Assume $\cos\alpha = c/d$ for some integers c and d with $d > 0$. Then the above implies that d times the left-hand side of (3) is an integer. On the other hand,

$$\left|d\int_0^\alpha f(x)\sin x\,dx\right| < d\alpha\frac{\alpha^{4n-2}b^{3n-1}}{(n-1)!} = d\alpha^3b^2\frac{(\alpha^4b^3)^{n-1}}{(n-1)!}. \tag{4}$$

The latter expression is < 1 for n sufficiently large. Thus far the proof has been quite similar to the proof of the Theorem 7; however, we're not done. The integral above can be negative (unlike the integral dealt with in the proof of Theorem 7). We get from (4) that

$$d \int_0^\alpha f(x) \sin x \, dx = 0$$

provided n is sufficiently large. We will obtain a contradiction by examining (3) somewhat closer. Recall that we used (3) to obtain that the left-hand side above is an integer; we will show that it is not divisible by n if we choose n appropriately. This then will contradict the fact that it is 0.

Recall in (3) that $F'(\alpha) = 0$. Observe that $f^{(k)}(0) = 0$ if $0 \leq k < n - 1$ and, from Lemma 1, $f^{(k)}(0)$ is divisible by n if $k > n - 1$. Also,

$$f^{(n-1)}(0) = a^{2n}(2a)^{n-1}.$$

We choose $n = p$ where p is a sufficiently large prime so that the integral in (3) is 0 as above and so that $p > \max\{2a, d\}$. Then we obtain that $f^{(k)}(0)$ is divisible by p if and only if $k \neq p-1$. Hence, $F(0)$ is not divisible by p. On the other hand, $f^{(k)}(\alpha) = 0$ if $0 \leq k < 2p - 1$, and it follows from Lemma 1 that $f^{(k)}(\alpha)$ is divisible by p for all non-negative integers k. Thus, $F(\alpha)$ is an integer divisible by p. After multiplying through by d in (3) and recalling that we chose $p > d$, we see that the left-hand side of (4) is an integer which is not divisible by p, and the proof is complete. □

It is worth noting that some of the difficulties could have been avoided if we were working with $\cosh(\alpha)$ instead of $\cos(\alpha)$. Replacing the roles of the trignometric functions above with the hyperbolic functions, we would obtain the analog to (4) with the integrand being $f(x)\sinh(x)$. Since the integrand would then be positive, the difficulty at the end of the proof would have been avoided. This would then lead easily to a proof that if α is rational, then e^α for $\alpha \neq 0$ and $\log \alpha$ for $\alpha \neq 1$ are irrational. Since we will shortly be proving these numbers transcendental, we do not labor on this point.

Corollary 1. *If α is a non-zero rational number, then the numbers $\cos(\alpha)$, $\sin(\alpha)$, $\tan(\alpha)$, $\sec(\alpha)$, $\csc(\alpha)$, and $\cot(\alpha)$ are all irrational. Furthermore, the squares of these numbers are irrational.*

Proof. It suffices to prove the second half of the corollary. This follows from expressing each of the values of the squares of the trignometric functions in terms of $\cos(\alpha)$. Or first observe that

$$\cos^2(\alpha) = (1 + \cos(2\alpha))/2$$

to establish the result for $\cos^2(\alpha)$, and then use that

$$\sin^2(\alpha) = 1 - \cos^2(\alpha)$$

and

$$\tan^2(\alpha) = \sec^2(\alpha) - 1 = \frac{1}{\cos^2(\alpha)} - 1$$

to obtain the result for $\sin^2(\alpha)$ and $\tan^2(\alpha)$. The others follow by considering reciprocals. □

Homework:

1. Prove that if α is a non-zero rational number, then cosh(α) is irrational. (You can probably find this in a book, but you should try simply modifying the argument we gave for $\cos(\alpha)$ being irrational. Recall that the second half of that argument should not be necessary for this problem.)