1 Introduction

Definition 1. *A rational number is a number which can be expressed in the form a/b where a and b are integers with $b > 0$.*

Theorem 1. *A real number α is a rational number if and only if it can be expressed as a repeating decimal, that is if and only if $\alpha = m.d_1 d_2 \ldots d_k \overline{d_{k+1} d_{k+2} \ldots d_{k+r}}$, where $m = [\alpha]$ if $\alpha \geq 0$ and $m = -[|\alpha|]$ if $\alpha < 0$, where k and r are non-negative integers with $r \geq 1$, and where the d_j are digits.*

Proof. If

$$\alpha = m.d_1 d_2 \ldots d_k \overline{d_{k+1} d_{k+2} \ldots d_{k+r}},$$

then $(10^{k+r} - 10^k)\alpha \in \mathbb{Z}$ and it easily follows that α is rational.

If $\alpha = a/b$ with a and b integers and $b > 0$, then $\alpha = m.d_1 d_2 \ldots$ for some digits d_j. If $\{x\}$ denotes the fractional part of x, then

$$\{10^j |\alpha|\} = 0.d_{j+1} d_{j+2} \ldots . \tag{1}$$

On the other hand,

$$\{10^j |\alpha|\} = \{10^j a/b\} = u/b \quad \text{for some } u \in \{0, 1, \ldots, b-1\}.$$

Hence, by the pigeon-hole principle, there exist non-negative integers k and r with $r \geq 1$ and

$$\{10^k |\alpha|\} = \{10^{k+r} |\alpha|\}.$$

From (1), we deduce that

$$0.d_{k+1} d_{k+2} \cdots = 0.d_{k+r+1} d_{k+r+2} \ldots$$

so that

$$\alpha = m.d_1 d_2 \ldots d_k \overline{d_{k+1} d_{k+2} \ldots d_{k+r}},$$

and the result follows. □

Definition 2. *A number is irrational if it is not rational.*

Theorem 2. *A real number α which can be expressed as a non-repeating decimal is irrational.*

Proof 1. From the argument above, if $\alpha = m.d_1 d_2 \ldots$ and $\alpha = a/b$ is rational, then the digits d_j repeat. This implies the desired result. □

Proof 2. This proof is based on showing that the decimal representation of a number is essentially unique. Assume α can be expressed as a non-repeating decimal and is rational. By Theorem 1, there are digits d_j and d'_j such that

$$\alpha = m.d_1 d_2 \ldots d_k \overline{d_{k+1} d_{k+2} \ldots d_{k+r}} \quad \text{and} \quad \alpha = m.d'_1 d'_2 d'_3 \ldots,$$

where the latter represents a non-repeating decimal. Then there is a minimum positive integer u such that $d_u \neq d'_u$. Observe that there must be a $v > u$ such that $|d_v - d'_v| \neq 9$; otherwise, we would have that $d'_v = 9 - d_v$ for every $v > u$, contradicting that the d'_v do not repeat. Hence,

$$\begin{aligned} 0 &= |\alpha - \alpha| \\ &= |m.d_1 d_2 \dots d_k \overline{d_{k+1} d_{k+2} \dots d_{k+r}} - m.d'_1 d'_2 d'_3 \dots | \\ &\geq \frac{|d_u - d'_u|}{10^u} - \sum_{j=u+1}^{\infty} \frac{|d_j - d'_j|}{10^j} \\ &> \frac{1}{10^u} - \sum_{j=u+1}^{\infty} \frac{9}{10^j}. \end{aligned}$$

The last expression is easily evaluated to be 0 (the series is a geometric series). Hence, we obtain a contradiction, which shows that α must be irrational. $\square$

We will begin the course by briefly discussing the irrationality of certain numbers, namely $\sqrt{2}$, $\log_{10} 2$, e, π, $\log 2$ (natural logarithm of 2), and $\zeta(3)$ (to be defined). It is nevertheless convenient to define now the main topic of this course.

Definition 3. *An algebraic number is a number which is a root of $f(x) \in \mathbb{Z}[x]$ for some $f(x) \not\equiv 0$. A transcendental number is a number which is not algebraic.*

It should be noted that rational numbers correspond to roots of linear polynomials in $\mathbb{Z}[x]$.

Examples of transcendental numbers include e, π, and e^π. The number $\sqrt{2}$ is an easy example of a number which is irrational but not transcendental.

There are many open problems concerning the subject. We do not know if the numbers $e + \pi$, $e\pi$, or π^e are transcendental. We know that $\log 2$ and $\log 3$ are transcendental, but we do not know if $(\log 2)(\log 3)$ is. Euler's constant is

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right);$$

we do not even know if it is irrational. The Riemann zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ (for $\mathcal{R}(s) > 1$). It is known that $\zeta(2n)$ is transcendental whenever n is a positive integer, but the status of $\zeta(2n+1)$ is not very well understood. In 1978, Apery gave the first proof that $\zeta(3)$ is irrational, and very recently it was established that $\zeta(2n+1)$ is irrational for infinitely many positive integers n.

We now turn to some irrationality examples.

Theorem 3. *If the real number α is a root of*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x],$$

then α is either an integer or an irrational number.

Proof. Prove directly or by using the rational root test. Suppose $\alpha = a/b$ with $b > 0$ and $(a, b) = 1$, and show that $b = 1$ (that is that b has no prime divisors). $\square$

Corollary 1. *If n and k are positive integers and n is not an kth power, then $\sqrt[k]{n}$ is irrational.*

Proof. Clear. □

Theorem 4. *There are real irrational numbers α and β for which α^β is rational.*

Proof. Either $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ is an example. (Also, after the next theorem, the example $\sqrt{10}^{2\log_{10} 2}$ leads to a simple argument here.) □

Theorem 5. *The number $\log_{10} 2$ is irrational.*

Proof. Trivial (use the Fundamental Theorem of Arithmetic). □

Theorem 6. *The number e is irrational.*

Proof. Almost trivial. □

Homework:

1. Justify the last sentence in the proof of Theorem 1. (Note that $0.1 = 0.0\overline{9}$.)

2. Let a and b be positive integers, and write $a/b = m.d_1d_2\dots d_k\overline{d_{k+1}d_{k+2}\dots d_{k+r}}$, where m is a positive integer, the d_j are digits, and r is chosen as small as possible. Prove that r divides $\phi(b)$ where ϕ is Euler's $\phi-$function.

3. From (1), it follows that $r \le b-1$ and that if $r = b-1$, then b is a prime (note: the converse of this isn't true). Suppose $r = b-1$.

   (i) Prove that each of the digits $0, 1, \dots, 9$ occurs among the digits $d_{k+1}, d_{k+2}, \dots, d_{k+r}$ either $[(b-1)/10]$ or $[(b-1)/10]+1$ times. (For example, $r = 46$ for $1/47$ and each of the digits $0, 3, 6, 9$ occurs 4 times in the "periodic part" of $1/47$ and each of the other digits occurs 5 times in the periodic part of $1/47$; and $r = 60$ for $1/61$, and it follows from this problem that each digit occurs exactly 6 times in the periodic part of $1/61$.)

   (ii) Prove that 0 occurs $[(b-1)/10]$ times among the digits $d_{k+1}, d_{k+2}, \dots, d_{k+r}$.

4. Using an argument similar to the proof of Theorem 6 (e is irrational), prove that e^2 is irrational.
