1 Introduction

Definition 1. A rational number is a number which can be expressed in the form \(a/b\) where \(a\) and \(b\) are integers with \(b > 0\).

Theorem 1. A real number \(\alpha\) is a rational number if and only if it can be expressed as a repeating decimal, that is if and only if \(\alpha = m.d_1d_2\ldots d_k\overline{d_{k+1}d_{k+2}\ldots d_{k+r}}\), where \(m = [\alpha]\) if \(\alpha \geq 0\) and \(m = -\lfloor|\alpha|\rfloor\) if \(\alpha < 0\), where \(k\) and \(r\) are non-negative integers with \(r \geq 1\), and where the \(d_j\) are digits.

Proof. If 
\[
\alpha = m.d_1d_2\ldots d_k\overline{d_{k+1}d_{k+2}\ldots d_{k+r}},
\]
then \((10^{k+r} - 10^k)\alpha \in \mathbb{Z}\) and it easily follows that \(\alpha\) is rational.

If \(\alpha = a/b\) with \(a\) and \(b\) integers and \(b > 0\), then \(\alpha = m.d_1d_2\ldots\) for some digits \(d_j\). If \(\{x\}\) denotes the fractional part of \(x\), then
\[
\{10^j|\alpha|\} = 0.d_{j+1}d_{j+2}\ldots
\]
On the other hand,
\[
\{10^j|\alpha|\} = \{10^j a/b\} = u/b\quad \text{for some } u \in \{0, 1, \ldots, b - 1\}.
\]
Hence, by the pigeon-hole principle, there exist non-negative integers \(k\) and \(r\) with \(r \geq 1\) and
\[
\{10^k|\alpha|\} = \{10^{k+r}|\alpha|\}.
\]
From (1), we deduce that
\[
0.d_{k+1}d_{k+2}\ldots = 0.d_{k+r+1}d_{k+r+2}\ldots
\]
so that
\[
\alpha = m.d_1d_2\ldots d_k\overline{d_{k+1}d_{k+2}\ldots d_{k+r}},
\]
and the result follows.

Definition 2. A number is irrational if it is not rational.

Theorem 2. A real number \(\alpha\) which can be expressed as a non-repeating decimal is irrational.

Proof 1. From the argument above, if \(\alpha = m.d_1d_2\ldots\) and \(\alpha = a/b\) is rational, then the digits \(d_j\) repeat. This implies the desired result.

Proof 2. This proof is based on showing that the decimal representation of a number is essentially unique. Assume \(\alpha\) can be expressed as a non-repeating decimal and is rational. By Theorem 1, there are digits \(d_j\) and \(d_j'\) such that
\[
\alpha = m.d_1d_2\ldots d_k\overline{d_{k+1}d_{k+2}\ldots d_{k+r}}\quad \text{and} \quad \alpha = m.d_1'd_2'd_3'\ldots,
\]
where the latter represents a non-repeating decimal. Then there is a minimum positive integer $u$ such that $d_u \neq d'_u$. Observe that there must be a $v > u$ such that $|d_v - d'_v| \neq 9$; otherwise, we would have that $d'_v = 9 - d_v$ for every $v > u$, contradicting that the $d'_v$ do not repeat. Hence,

$$0 = |\alpha - \alpha| = |m.d_1 d_2 \ldots d_k d_{k+1} d_{k+2} \ldots d_{k+r} - m.d'_1 d'_2 d'_3 \ldots| \geq \frac{|d_u - d'_u|}{10^u} - \sum_{j=u+1}^{\infty} \frac{|d_j - d'_j|}{10^j} > \frac{1}{10^u} - \sum_{j=u+1}^{\infty} \frac{9}{10^j}.$$  

The last expression is easily evaluated to be 0 (the series is a geometric series). Hence, we obtain a contradiction, which shows that $\alpha$ must be irrational.  

We will begin the course by briefly discussing the irrationality of certain numbers, namely $\sqrt{2}$, $\log_{10} 2$, $e$, $\pi$, $\log 2$ (natural logarithm of 2), and $\zeta(3)$ (to be defined). It is nevertheless convenient to define now the main topic of this course.

**Definition 3.** An algebraic number is a number which is a root of $f(x) \in \mathbb{Z}[x]$ for some $f(x) \neq 0$. A transcendental number is a number which is not algebraic.

It should be noted that rational numbers correspond to roots of linear polynomials in $\mathbb{Z}[x]$.

Examples of transcendental numbers include $e$, $\pi$, and $e^\pi$. The number $\sqrt{2}$ is an easy example of a number which is irrational but not transcendental.

There are many open problems concerning the subject. We do not know if the numbers $e + \pi$, $e\pi$, or $\pi^e$ are transcendental. We know that $\log 2$ and $\log 3$ are transcendental, but we do not know if $(\log 2)(\log 3)$ is. Euler’s constant is

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right);$$

we do not even know if it is irrational. The Riemann zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ (for $\Re(s) > 1$). It is known that $\zeta(2n)$ is transcendental whenever $n$ is a positive integer, but the status of $\zeta(2n+1)$ is not very well understood. In 1978, Apery gave the first proof that $\zeta(3)$ is irrational, and very recently it was established that $\zeta(2n+1)$ is irrational for infinitely many positive integers $n$.

We now turn to some irrationality examples.

**Theorem 3.** If the real number $\alpha$ is a root of

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x],$$

then $\alpha$ is either an integer or an irrational number.

**Proof.** Prove directly or by using the rational root test. Suppose $\alpha = a/b$ with $b > 0$ and $(a, b) = 1$, and show that $b = 1$ (that is that $b$ has no prime divisors).  

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Corollary 1. If \( n \) and \( k \) are positive integers and \( n \) is not an \( k \)th power, then \( \sqrt[k]{n} \) is irrational.

Proof. Clear.

Theorem 4. There are real irrational numbers \( \alpha \) and \( \beta \) for which \( \alpha^\beta \) is rational.

Proof. Either \( \sqrt{2}^{\sqrt{2}} \) or \( (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \) is an example. (Also, after the next theorem, the example \( \sqrt[10]{2\log_{10} 2} \) leads to a simple argument here.)

Theorem 5. The number \( \log_{10} 2 \) is irrational.

Proof. Trivial (use the Fundamental Theorem of Arithmetic).

Theorem 6. The number \( e \) is irrational.

Proof. Almost trivial.

Homework:

1. Justify the last sentence in the proof of Theorem 1. (Note that 0.1 = 0.0\(\overline{5}\).)

2. Let \( a \) and \( b \) be positive integers, and write \( a/b = m.d_1d_2 \ldots d_kd_{k+1}d_{k+2} \ldots d_{k+r} \), where \( m \) is a positive integer, the \( d_j \) are digits, and \( r \) is chosen as small as possible. Prove that \( r \) divides \( \phi(b) \) where \( \phi \) is Euler’s \( \phi \) function.

3. From (1), it follows that \( r \leq b - 1 \) and that if \( r = b - 1 \), then \( b \) is a prime (note: the converse of this isn’t true). Suppose \( r = b - 1 \).

   (i) Prove that each of the digits 0, 1, \ldots, 9 occurs among the digits \( d_{k+1}, d_{k+2}, \ldots, d_{k+r} \) either \( \lfloor (b - 1)/10 \rfloor \) or \( \lceil (b - 1)/10 \rceil + 1 \) times. (For example, \( r = 46 \) for 1/47 and each of the digits 0, 3, 6, 9 occurs 4 times in the “periodic part” of 1/47 and each of the other digits occurs 5 times in the periodic part of 1/47; and \( r = 60 \) for 1/61, and it follows from this problem that each digit occurs exactly 6 times in the periodic part of 1/61.)

   (ii) Prove that 0 occurs \( \lfloor (b - 1)/10 \rfloor \) times among the digits \( d_{k+1}, d_{k+2}, \ldots, d_{k+r} \).

4. Using an argument similar to the proof of Theorem 6 (\( e \) is irrational), prove that \( e^2 \) is irrational.