MATH 574: TEST 2

Name ______________________________

Instructions and Point Values: Put your name in the space provided above. Make sure that your test has six different pages including one blank page. Justify ALL answers with the proper work. Calculators are NOT permitted on this test.

Point Values: Problem 7 is worth 16 points. Each remaining problem is worth 12 points.

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(1) How many subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have exactly 3 elements?

Answer: $\boxed{}$ (simplify your answer)

(2) How many permutations of $\{a, b, c, d, e\}$ have the letters a and b coming somewhere before c? For example, bdace is one such permutation.

Answer: $\boxed{}$ (simplify your answer)

(3) How many 4-permutations of the set $\{1, 2, 3, 4, 5\}$ contain the number 3?

Answer: $\boxed{}$ (simplify your answer)

(4) How many 4 digit numbers are there having the property that no two consecutive digits are even and no two consecutive digits are odd? For example, 2741 and 9636 are two such numbers. (The leading digit is not to be 0.)

Answer: $\boxed{}$ (you do not need to simplify your answer)

(5) Let S be a set of 20 points such that no three of them are collinear (no line passes through three of the points). Let T be the set of all lines passing through two points in S. How many lines are in T?

Answer: $\boxed{}$ (simplify your answer)

(6) If 3^k is the largest power of 3 dividing 200!, then what is the value of k?

Answer: $\boxed{}$ (simplify your answer)

(7) (a) Explain why

$$\binom{10}{0}+2\binom{10}{1}+2^2\binom{10}{2}+\cdots+2^8\binom{10}{8}+2^9\binom{10}{9}+2^{10}\binom{10}{10}=3^{10}.$$

(b) The value of

$$2^{10}\binom{10}{0}+2^9\binom{10}{1}+2^8\binom{10}{2}+\cdots+2^2\binom{10}{8}+2\binom{10}{9}+\binom{10}{10}$$

is also equal to a^b for some integers a and b both > 1. What are a and b? Justify your answers (I am not looking for something complicated here – a good justification can be concise).

$a =$ $\boxed{}$ $\qquad$ $b =$ $\boxed{}$

(8) Let k_1, k_2, k_3, ℓ_1, ℓ_2, and ℓ_3 be positive integers and $g(x)$ be a polynomial satisfying

$$(*) \qquad (x-1)^3 g(x) = x^{k_1} + x^{k_2} + x^{k_3} - x^{\ell_1} - x^{\ell_2} - x^{\ell_3}.$$

Show that

$$(**) \qquad k_1 + k_2 + k_3 = \ell_1 + \ell_2 + \ell_3 \qquad \text{and} \qquad k_1^2 + k_2^2 + k_3^2 = \ell_1^2 + \ell_2^2 + \ell_3^2.$$

For example, since

$$(x-1)^3\left(x^7 + 3x^6 + 4x^5 + 4x^4 + 3x^3 + 2x^2 + x\right) = x^{10} + x^5 + x^2 - x^8 - x^8 - x^1,$$

we can deduce that

$$10 + 5 + 2 = 8 + 8 + 1 \quad \text{and} \quad 10^2 + 5^2 + 2^2 = 8^2 + 8^2 + 1^2.$$

The problem is to show that in general when $(*)$ holds, then $(**)$ holds.

Hint: Recall the arguments for evaluating the sums $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}$ and $\displaystyle\sum_{k=0}^{n} k^2\binom{n}{k}$.