MATH 574: FINAL EXAM

Name ______________________________

Instructions and Point Values: Put your name in the space provided above. Check that your test contains 14 different pages including one blank page. Work each problem below and show ALL of your work. Do NOT use a calculator.

There are 285 total points possible on this exam. The point total for each problem in each part is indicated below.

PART I

Problem (1) is worth 20 points.

Problem (2) is worth 20 points.

Problem (3) is worth 20 points.

Problem (4) is worth 15 points.

Problem (5) is worth 15 points.

Problem (6) is worth 15 points.

Problem (7) is worth 20 points.

Problem (8) is worth 20 points.

PART II

Problem (1) is worth 20 points.

Problem (2) is worth 20 points.

Problem (3) is worth 20 points.

Problem (4) is worth 20 points.

Problem (5) is worth 20 points.

Problem (6) is worth 20 points.

Problem (7) is worth 20 points.

EXTRA CREDIT

10 points

PART I: Problems You've Seen (SHOW WORK!!)

(1) Prove that $\log_2 3$ is irrational.

(2) Using induction, prove that the sum of the first n odd numbers is n^2. (Note: I want a proof by induction even though it is possible to give other proofs for this result.)

(3) The Fibonacci numbers f_n are defined by the recursion $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Prove that for every integer $n \geq 1$, the number f_{3n} is even and the numbers f_{3n-1} and f_{3n-2} are odd.

(4) A person has 30 socks in a drawer, 15 of which are blue and 15 of which are brown. How many socks must he pull out to assure he has two socks the same color? Justify your answer.

Answer: $\boxed{}$

(5) How many subsets does an n-element set have? Justify your answer.

Answer: $\boxed{}$

(6) Consider the positive integers ≤ 200. How many are divisible by 2, 3, or 5? Show your work.

Answer: $\boxed{}$

(7) Calculate $\sum_{k=0}^{n} k\binom{n}{k}$ in closed form. Show how the answer is derived (explain where the answer comes from).

Answer: $\boxed{}$

(8) How many solutions are there to the equation

$$x_1 + x_2 + x_3 = 20$$

if each x_j is to be an integer from $\{0, 1, 2, \ldots, 20\}$? Show how the answer is derived. Don't just give a formula; if you want to use a formula, explain where the formula comes from.

Answer: $\boxed{}$

PART II.

(1) Two people play a game. They begin with $N = 0$. Each person takes turns choosing a number from $\{1, 2, 3, \ldots, 8\}$, adding it to N to form a new number N, and announcing what the new N is. The winner is the first person to get the number N to be ≥ 200.

(a) Is it better to move first or second in this game? (Which one?) Show work.

(b) If the first player begins by choosing the number 7 (so $N = 7$), then what is the best number from $\{1, 2, 3, \ldots, 8\}$ for the second player to choose?

(2) A lattice point in space is a point (x, y, z) with each of x, y, and z being integers. For example, $(3, 0, -2)$ is a lattice point. Prove that if there are nine or more lattice points in space, then the midpoint of some two of them is a lattice point.

Hint: If (x_1, y_1, z_1) and (x_2, y_2, z_2) are lattice points, then the midpoint is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

(3) In this problem, I am testing you on certain terminology associated with graphs. In addition, I am testing to see how well you can comprehend the following definition:

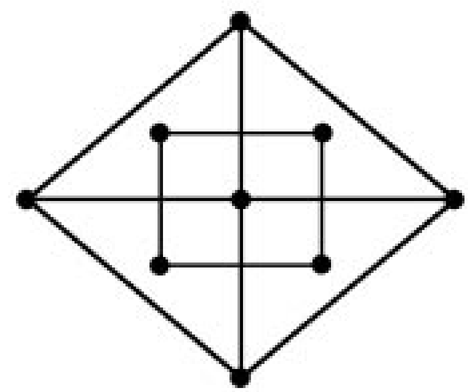
The *diameter* of a graph G is the maximum distance between two vertices of G.

Circle the correct answer to the following questions and give a short justification of the answer. Each question refers to the graph to the right of the question.

Is this graph complete? **YES** **NO**

Is this graph connected? **YES** **NO**

Is this graph planar? **YES** **NO**



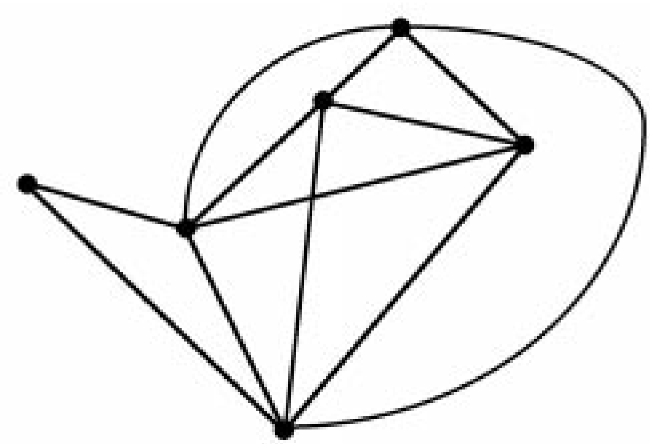
Is this graph complete? **YES** **NO**

Is this graph connected? **YES** **NO**

Is this graph planar? **YES** **NO**

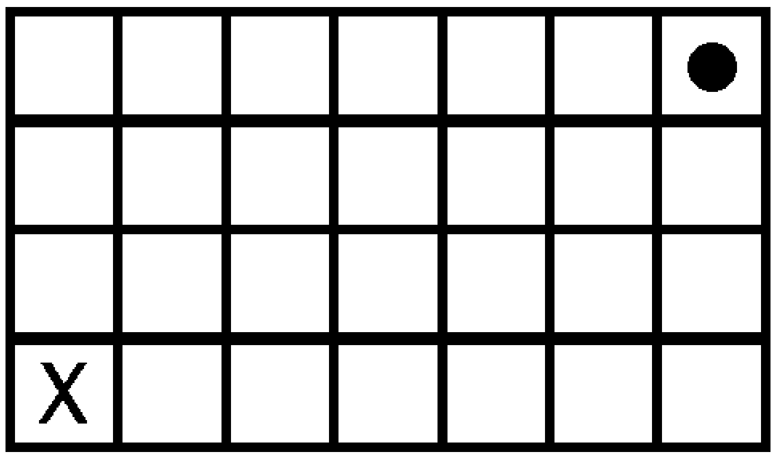
What is the diameter of this graph?

1 **2** **3** **4** **5** **6** (circle one)



(4) Six people get together and some handshakes take place. Explain why there must either be three people any two of which shook hands with each other or three people any two of which did not shake hands with each other.

(5) Two players play a game on the board below as follows. Each person takes turns moving the shaded circle either downward at least one square or to the left at least one square (so each turn consists of moving either downward or to the left but not both). The first person to place the shaded circle on the square marked with and X wins. How should the first player begin this game? Indicate the best first move by putting the letter A on the appropriate square (where the first player should put the shaded circle). Justify your answer (explain why the best first move is in fact the best first move).



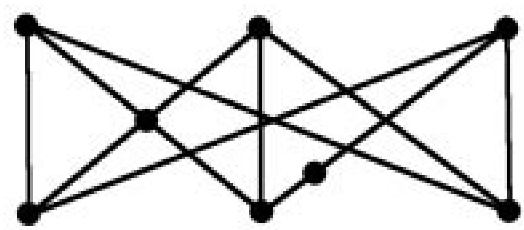
(6) At a party, various people shake hands. Explain why there must be an even number of people who have shaken an odd number of hands at the party.

Hint: Suppose n is the number of people shaking hands, and let d_j denote the number of hands the jth person shook. Justify first that $d_1 + d_2 + \cdots + d_n$ is even, and then explain why that implies an even number of the d_j's are odd.

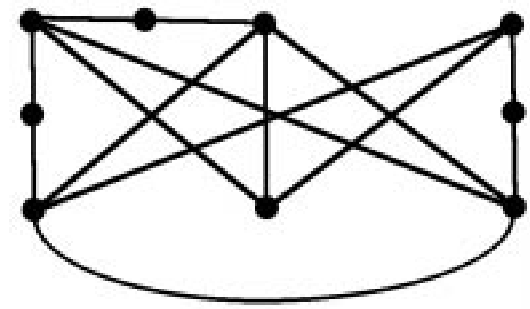
(7) Let A_n be the number of ways of covering a $2 \times n$ board using red 2×1 boards, blue 2×1 boards, green 2×1 boards, purple 1×2 boards, and yellow 1×2 boards. Thus, there are 3 ways of covering a 2×1 board (so $A_1 = 3$), namely by using one red 2×1 board or one blue 2×1 board or one green 2×1 board. Also, there are 13 ways to cover a 2×2 board (so $A_2 = 13$). For example, you could use a red 2×1 board followed by a green 2×1 board, or a green 2×1 board followed by a red 2×1 board, or a red 2×1 board followed by a red 2×1 board, or a purple 1×2 board above a yellow 1×2 board, or a yellow 1×2 board above a purple 1×2 board. That accounts for 5 of the 13 different coverings of a 2×2 board. Give an explicit formula for A_n in terms of n. In other words, find a recursion relationship that A_n satisfies and solve it.

EXTRA CREDIT

Despite what you may have thought you heard in class, exactly one of the following graphs is planar.



Graph G



Graph H

(a) Clarify which graph is planar?

Answer: $\boxed{}$ (Graph G or Graph H)

(b) Justify your answer to (a) (i.e., explain why the answer you gave is in fact planar).

(c) Explain what significant difference exists in the graphs above that makes your answer correct (what about your answer to (a) makes it planar that is not true about the other graph). I am not looking for a definition of planar here. This is extra credit, and if you do not give the answer that is "most appropriate", then you will not get credit for this part.