MATH 574: TEST 1

Name Solutions

Points: Problems (1), (2), and (3) are worth 15 points each, and Problems (4), (5), and (6) are worth 18 points each. Your correctly spelled name or nickname above is worth 1 point.

(1) Suppose we want to use induction to prove

$$8^n - 7n - 1 \text{ is divisible by } 49$$

for all positive integers n.

(a) What would be the "induction hypothesis" in the proof? (Don't forget to use the word "some" or the word "all" below.)

Suppose that $8^k - 7k - 1$ is divisible by 49 for some positive integer k.

(b) After stating the induction hypothesis in the proof, what should the goal be? In other words, what should we be trying to establish? Be precise.

The goal would then be to show that $8^{k+1} - 7(k+1) - 1$ is divisible by 49 (for the same k as in part (a)).

(2) Prove that $\log_2 3$ is irrational.

Proof. Assume $\log_2 3$ is rational. Then there are positive integers a and b such that $\log_2 3 = a/b$. This implies that $3 = 2^{a/b}$. Hence,

$$3^b = \left(2^{a/b}\right)^b = 2^a.$$

Since the right-hand side is even and the left-hand side is odd, we get a contradiction. Therefore, $\log_2 3$ is irrational. ■

(3) We showed in class that $\sqrt{2}$ is irrational. Using this information, complete the boxes below so as to complete a proof that $\sqrt{17+\sqrt{2}}$ is irrational.

Proof. | Assume $\sqrt{17+\sqrt{2}}$ is rational. |

(Put the first sentence of the proof above.)

Then there are integers a and b, with $b > 0$, such that | $\sqrt{17+\sqrt{2}} = \frac{a}{b}$ |.

It follows that $17+\sqrt{2} =$ | a^2/b^2 |. Hence,

$$\sqrt{2} = \boxed{\frac{a^2}{b^2} - 17 = \frac{a^2 - 17b^2}{b^2}}.$$

(End up with something simplified here that justifies the next sentence.)

This is a contradiction since we showed in class that $\sqrt{2}$ is irrational. Therefore, $\sqrt{17+\sqrt{2}}$ is irrational. ■

(4) Let $a_1 = \sqrt{20}$, and let

$$a_{n+1} = \sqrt{20 + a_n} \quad \text{for each positive integer } n.$$

For example, $a_2 = \sqrt{20 + \sqrt{20}}$ and $a_3 = \sqrt{20 + \sqrt{20 + \sqrt{20}}}$. Complete the boxes below to give a proof that $a_n \leq 5$ for every positive integer n.

Proof. We prove

(∗) $$a_n \leq 5$$

for every positive integer n by [induction on n]. We observe first that [$\sqrt{20} \leq 5$] which implies (∗) holds for $n =$ [1]. Next, we make our induction hypothesis. We suppose that

(∗∗) [$a_k \leq 5$ for some positive integer k.]

(Do NOT refer to (∗). Also, use the word "all" *or* the word "some" above.)

Next, we show that (∗) holds when $n =$ [$k+1$]. In addition to (∗∗), we make use of [$a_{k+1} = \sqrt{20 + a_k}$] to obtain

(This is related to the statement of the problem.)

$$\boxed{a_{k+1}} = \boxed{\sqrt{20 + a_k}} \leq \boxed{\sqrt{20 + 5}} = \sqrt{25} = 5.$$

↑ (Use (∗∗) here.)

Therefore, (∗) holds for $n =$ [$k+1$]. Thus, (∗) holds for all positive integers n by [induction on n]. ■

(5) Let $k_1, k_2, k_3, \ldots, k_{98}, k_{99}$ be some ordering of the numbers $1, 2, 3, \ldots, 98, 99$. (For example, maybe $k_1 = 2$, $k_2 = 1$, and $k_j = 102 - j$ for $3 \leq j \leq 99$, but maybe not. That's just an example.) Complete the boxes below and finish the proof to show that there must be some positive integer $i \leq 99$ for which $k_i - i$ is even.

Proof. Assume that there is no positive integer $i \leq 99$ for which $k_i - i$ is even. Then the numbers $k_1 - 1, k_2 - 2, k_3 - 3, \ldots, k_{98} - 98$, and $k_{99} - 99$ are all odd. Since the sum of an odd number of odd numbers is [odd], we deduce that

$(k_1 - 1) + (k_2 - 2) + (k_3 - 3) + \cdots + (k_{98} - 98) + (k_{99} - 99)$ is [odd].

(A word goes here, not a number.)

Since $k_1, k_2, k_3, \ldots, k_{98}, k_{99}$ is some ordering of the numbers $1, 2, 3, \ldots, 98, 99$, we obtain

$$k_1 + k_2 + k_3 + \cdots + k_{98} + k_{99} = \boxed{1 + 2 + 3 + \cdots + 98 + 99}.$$

(You do not need an exact value here.)

Therefore,

$$\begin{aligned}
&(k_1 - 1) + (k_2 - 2) + (k_3 - 3) + \cdots + (k_{98} - 98) + (k_{99} - 99) \\
&= (k_1 + k_2 + k_3 + \cdots + k_{98} + k_{99}) - \left(\boxed{1 + 2 + 3 + \cdots + 98 + 99}\right) \\
&= \boxed{0}.
\end{aligned}$$

(An exact value here would be good.)

Finish the Proof.

Since 0 is an even number, we obtain a contradiction (a number cannot be both odd and even). Thus, our assumption is wrong, and we deduce that there must be some positive integer $i \leq 99$ for which $k_i - i$ is even. ■

(6) Let

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{1}, \quad a_3 = 1 + \cfrac{1}{1 + \cfrac{1}{1}}, \quad a_4 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}, \ldots.$$

Thus, $a_1 = 1$, $a_2 = 2$, $a_3 = 3/2$, $a_4 = 5/3$, and in general $a_n = 1 + (1/a_{n-1})$ for each integer $n \geq 2$. Complete the proof below to show that

$$(*) \qquad a_n = \frac{f_{n+1}}{f_n}$$

holds for every positive integer n where f_n is the n^{th} Fibonacci number. Recall that $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for every integer $n \geq 1$. Fill in the boxes below appropriately. Do **NOT** refer to $(*)$ in any of the boxes. Note that a proof should consist of complete English sentences.

Proof. We prove $(*)$ holds for every positive integer n by [induction on n].

First, we verify that $(*)$ holds when $n =$ [1]. For this value of n, $(*)$ is true since [$f_2 = f_1$ so $a_1 = 1 = f_2/f_1$]. Now, we make our induction hypothesis. We suppose that

$(**)$ [$a_k = f_{k+1}/f_k$ for some positive integer k.]

(Do **NOT** refer to $(*)$. Also, what you write should make sense with the rest of the argument.)

Next, we consider the case $n =$ [$k+1$]. From $(**)$, we deduce that

$$a_{k+1} = 1 + \frac{1}{a_k} = 1 + \frac{1}{\boxed{f_{k+1}/f_k}} = \frac{\boxed{f_{k+1}} + \boxed{f_k}}{\boxed{f_{k+1}}}.$$

Since [$f_{k+2} = f_{k+1} + f_k$], we obtain that $a_{k+1} = f_{k+2}/f_{k+1}$ so that

$(*)$ holds for $n = k+1$. This proves $(*)$ holds for every positive integer n by induction. ■