ANSWERS TO SPRING, 1999, TEST 2

1. $\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}$

2. $10\cdot 9\cdot 8=720$

3. $500+333-166=667$

4. $\binom{6}{3}\binom{6}{2}=20\cdot 15=300$

5. $10!-9!=9\cdot 9!$

6. (a) $(x+y)^n=\displaystyle\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$ where n is a positive integer

   (b) 2^n (take $x=y=1$)

   (c) 0 for n a positive integer (take $x=-1$ and $y=1$)

   (d) Take $n=100$ in parts (b) and (c) and add the results to get $2\displaystyle\sum_{j=0}^{50}\binom{100}{2j}=2^{100}$. Therefore, the answer is 2^{99}, that is $a=1$, $b=2$ and $c=99$.

7. The binomial theorem implies

$$(x+1)^n=\sum_{k=0}^{n}\binom{n}{k}x^k. \tag{1}$$

Integrating the left side of (1) from $x=0$ to $x=1$, we get

$$\int_0^1(x+1)^n\,dx=\left.\frac{(x+1)^{n+1}}{n+1}\right|_0^1=\frac{2^{n+1}-1}{n+1}.$$

Integrating the right side of (1) from $x=0$ to $x=1$, we obtain

$$\int_0^1\sum_{k=0}^{n}\binom{n}{k}x^k\,dx=\sum_{k=0}^{n}\binom{n}{k}\left.\frac{x^{k+1}}{k+1}\right|_0^1=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}.$$

Hence,

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{k+1}=\frac{2^{n+1}-1}{n+1}.$$