ANSWERS TO FALL, 1999, TEST 2

1. $\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3!} = 56$

2. $4 + 6 \cdot 2 + 4! = 4 + 12 + 24 = 40$

3. $5! - 4! = 120 - 24 = 96$

4. $4 \cdot 5^3 + 5^4 = 1125$

5. $\binom{20}{2} = 190$

6. $\left\lfloor \frac{200}{3} \right\rfloor + \left\lfloor \frac{200}{3^2} \right\rfloor + \left\lfloor \frac{200}{3^3} \right\rfloor + \cdots = 66 + 22 + 7 + 2 = 97$

7. (a) The binomial theorem implies

   $$(x+y)^{10} = \binom{10}{0} y^{10} + \binom{10}{1} xy^9 + \binom{10}{2} x^2y^8 + \cdots + \binom{10}{8} x^8y^2 + \binom{10}{9} x^9y + \binom{10}{10} x^{10}.$$

   Take $x = 2$ and $y = 1$.

   (b) Take $x = 1$ and $y = 2$. This gives $a = 3$ and $b = 10$ (though there are other possibilities).

8. Let $F(x) = (x-1)^3 g(x)$. Then

   $$F'(x) = 3(x-1)^2 g(x) + (x-1)^3 g'(x) = (x-1)^2 h(x),$$

   where $h(x) = 3g(x) + (x-1)g'(x)$. Then

   $$F''(x) = 2(x-1)h(x) + (x-1)^2 h'(x).$$

   Hence, $F'(1) = F''(1) = 0$. Given $(*)$, we also have

   $$F(x) = x^{k_1} + x^{k_2} + x^{k_3} - x^{\ell_1} - x^{\ell_2} - x^{\ell_3},$$

   $$F'(x) = k_1 x^{k_1-1} + k_2 x^{k_2-1} + k_3 x^{k_3-1} - \ell_1 x^{\ell_1-1} - \ell_2 x^{\ell_2-1} - \ell_3 x^{\ell_3-1},$$

   $$F''(x) = k_1(k_1-1)x^{k_1-2} + k_2(k_2-1)x^{k_2-2} + k_3(k_3-1)x^{k_3-2} - \ell_1(\ell_1-1)x^{\ell_1-2} - \ell_2(\ell_2-1)x^{\ell_2-2} - \ell_3(\ell_3-1)x^{\ell_3-2}.$$

   Using $F'(1) = 0$, we obtain the first equation in $(**)$. Using $F'(1) + F''(1) = 0$, we obtain the second equation in $(**)$.