MATH 555 NOTES
Integration by Parts Formula for Riemann-Stieltjes Integrals

In this section, we discuss integration by parts. This is a useful result for Riemann-Stieltjes integrals, but we note that in practice often an application of Abel summation (which we do not discuss here) can be used instead of integration by parts.

Theorem. *Let f and g be increasing real-valued functions on $[a,b]$. If $g \in \mathcal{R}_f[a,b]$, then $f \in \mathcal{R}_g[a,b]$ and*

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df.$$

Proof. For any partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$, we have

$$\begin{aligned} U(\mathcal{P},f,g) &= \sum_{j=1}^n f(x_j)\big(g(x_j) - g(x_{j-1})\big) = \sum_{j=1}^n f(x_j)g(x_j) - \sum_{j=1}^n f(x_j)g(x_{j-1}) \\ &= f(x_n)g(x_n) - f(x_0)g(x_0) + \sum_{j=1}^n f(x_{j-1})g(x_{j-1}) - \sum_{j=1}^n f(x_j)g(x_{j-1}) \\ &= f(x_n)g(x_n) - f(x_0)g(x_0) - \sum_{j=1}^n g(x_{j-1})\big(f(x_j) - f(x_{j-1})\big) \\ &= f(b)g(b) - f(a)g(a) - L(\mathcal{P},g,f). \end{aligned}$$

Similarly,

$$U(\mathcal{P},g,f) = f(b)g(b) - f(a)g(a) - L(\mathcal{P},f,g).$$

Hence,

$$U(\mathcal{P},f,g) - L(\mathcal{P},f,g) = U(\mathcal{P},g,f) - L(\mathcal{P},g,f).$$

It follows that $f \in \mathcal{R}_g[a,b]$ (the right-hand side can be made $< \varepsilon$ for any fixed $\varepsilon > 0$ and so too then can the left-hand side).

Fix $\varepsilon > 0$. Then there is a partition $\mathcal{P}$ of $[a,b]$ such that

$$(*) \qquad L(\mathcal{P},g,f) > \int_a^b g\,df - \varepsilon \quad \text{and} \quad U(\mathcal{P},g,f) < \int_a^b g\,df + \varepsilon.$$

Hence,

$$\begin{aligned} f(b)g(b) - f(a)g(a) - \int_a^b g\,df - \varepsilon &< f(b)g(b) - f(a)g(a) - U(\mathcal{P},g,f) = L(\mathcal{P},f,g) \\ &\le \underline{\int_a^b} f\,dg = \int_a^b f\,dg = \overline{\int_a^b} f\,dg \le U(\mathcal{P},f,g) \\ &= f(b)g(b) - f(a)g(a) - L(\mathcal{P},g,f) < f(b)g(b) - f(a)g(a) - \int_a^b g\,df + \varepsilon. \end{aligned}$$

Since the above holds for every $\varepsilon > 0$, the theorem follows. ■

Questions: Why is there a partition $\mathcal{P}$ such that both inequalities simultaneously hold in $(*)$? Is it true that if f and g are increasing real-valued functions on $[a, b]$, then $g \in \mathcal{R}_f[a, b]$ if and only if $f \in \mathcal{R}_g[a, b]$?

Example: Estimate $\sum_{n \le x} \frac{1}{n}$. We take $f(t) = 1/t$ and $g(t) = [t]$. To use integration by parts as we have described it, we need f increasing; but our arguments below can be adjusted by taking $f(t) = -1/t$ which is increasing. One obtains

$$\sum_{n \le x} \frac{1}{n} = \int_{1/2}^{x} f\, dg = \int_1^x \frac{[t]}{t^2}\, dt + E_1 = \int_1^x \frac{1}{t}\, dt + E_2 = \log x + E_2,$$

where each E_j has absolute value ≤ 1 (for $x \ge 1$). Thus, $\sum_{n \le x} 1/n$ is close to $\log x$.

Homework:

(1) Prove that $\left(\sum_{n \le x} \frac{1}{n} \right) - \log x$ approaches a constant as x tends to infinity. (This constant is known as Euler's constant.)

(2) Show that $\sum_{2 \le n \le x} \frac{1}{n \log n} = \log\log x + E$ where $|E| \le 5$.

(3) Let $\pi(x)$ denote the number of primes $\le x$. It is known that $\pi(x) \le 6x/\log x$ for all $x \ge 2$. Prove that

$$\sum_{\substack{p \le x \\ p \text{ prime}}} \frac{1}{p} \le 10 \log\log x$$

for x sufficiently large.