MATH 555 NOTES
The Definition of the Riemann-Stieltjes Integral

Definitions and Notations. Let $a < b$, and let f and g be real valued functions on $[a, b]$ with f bounded and g increasing. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ (with $x_0 = a$, $x_n = b$, and $x_0 < x_1 < \cdots < x_n$). Let

$$M_j(f) = \sup\{f(t) : t \in [x_j, x_{j+1}]\} \quad \text{and} \quad m_j(f) = \inf\{f(t) : t \in [x_j, x_{j+1}]\}.$$

Define the *upper Riemann-Stieltjes sum* and *lower Riemann-Stieltjes sum* of f with respect to g by

$$U(\mathcal{P}, f, g) = \sum_{j=1}^{n} M_j(f)(g(x_j) - g(x_{j-1})) \quad \text{and} \quad L(\mathcal{P}, f, g) = \sum_{j=1}^{n} m_j(f)(g(x_j) - g(x_{j-1})),$$

respectively. Define the *upper Riemann-Stieltjes integral* and *lower Riemann-Stieltjes integral* of f with respect to g by

$$\overline{\int_a^b} f\, dg = \inf\{U(\mathcal{P}, f, g) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

and

$$\underline{\int_a^b} f\, dg = \sup\{L(\mathcal{P}, f, g) : \mathcal{P} \text{ is a partition of } [a, b]\},$$

respectively. Then we say that f *is Riemann-Stieltjes integrable (or integrable) with respect to g on $[a, b]$* if $\overline{\int_a^b} f\, dg = \underline{\int_a^b} f\, dg$. We call this common value of the upper and lower integrals the *Riemann-Stieltjes integral of f with respect to g on* $[a, b]$, and we denote it by $\int_a^b f\, dg$ or $\int_a^b f(x)\, dg(x)$. We let $\mathcal{R}_g[a, b]$ denote the set of all Riemann-Stieltjes integrable functions with respect to g on $[a, b]$.

Questions: Why do the upper and lower integrals exist above (why are they finite)? How does this definition differ from the definition of the Riemann integral.

Examples: (Explain intuitively.)

(1) If $f(x) = 1/x$ and $g(x) = [x]$, then for every positive integer n, we have

$$\int_1^n f\, dg = \sum_{k=2}^{n} \frac{1}{k} \qquad \text{and} \qquad \int_{1/2}^n f\, dg = \sum_{k=1}^{n} \frac{1}{k}.$$

(2) More generally, if $f(x)$ is a bounded function on $[1/2, n]$ that is continuous at each integer in $[1/2, n]$ and $g(x) = [x]$, then for every positive integer n, we have

$$\int_{1/2}^{n} f\,dg = \sum_{k=1}^{n} f(k).$$

(3) If

$$f(x) = x \qquad \text{and} \qquad g(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 < x < 2 \\ x^2 & \text{for } 2 \le x \le 3 \end{cases},$$

then

$$\int_0^3 f\,dg = \int_0^1 x(2x)\,dx + 2 \times 3 + \int_2^3 x(2x)\,dx = \frac{2}{3} + 6 + \frac{2}{3}(3^3 - 2^3) = \frac{58}{3}.$$

Many of the results for Riemann integrals and Riemann-Stieltjes integrals are essentially the same and can be established in analogous manners. For example, the following are true.

Theorem 1. *Let $a < b$, and let f and g be real valued functions on $[a, b]$ with f bounded and g increasing. Then the following all hold.*

(i) If $\mathcal{P}$ is a partition of $[a, b]$, then $L(\mathcal{P}, f, g) \le U(\mathcal{P}, f, g)$.

(ii) If $\mathcal{P}$ is a partition of $[a, b]$ and $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then $L(\mathcal{P}, f, g) \le L(\mathcal{Q}, f, g) \le U(\mathcal{Q}, f, g) \le U(\mathcal{P}, f, g)$.

(iii) If $\mathcal{P}$ and $\mathcal{Q}$ are arbitrary partitions of $[a, b]$, then $L(\mathcal{P}, f, g) \le U(\mathcal{Q}, f, g)$.

(iv) $\underline{\int_a^b} f\,dg \le \overline{\int_a^b} f\,dg.$

Theorem 2. *Let $a < b$, and let f and g be real valued functions on $[a, b]$ with f bounded and g increasing. Then $f \in \mathcal{R}_g[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}, f, g) - L(\mathcal{P}, f, g) < \varepsilon$.*

Homework:

(1) Prove Theorem 1.

(2) Prove Theorem 2.

(3) If we remove the requirement that g is monotonic in the definition of the Riemann-Stieltjes integral, we can still make sense out of the definition as is (the lower and upper integrals with respect to g no longer have to be finite, but in many instances they will be). Using an intuitive approach to such a definition, evaluate the following Riemann-Stieltjes integrals.

(a) $\displaystyle\int_0^{\pi} 1\,dg$ where $g(x) = \begin{cases} 2 & \text{if } x = \pi/2 \\ \sin x & \text{otherwise} \end{cases}$

(b) $\displaystyle\int_0^4 |x-2|\,d[x]$

(c) $\displaystyle\int_0^4 [x]\,d|x-2|$