# Axioms for a Finite Projective Plane of Order $n$

**Axiom P1.** There exist at least 4 distinct points no 3 of which are collinear.

**Axiom P2.** There exists at least 1 line with exactly $n+1$ points on it.

**Axiom P3.** Given any 2 distinct points, there exists exactly one line passing through the 2 points.

**Axiom P4.** Given any two distinct lines, there exists at least one point where the lines intersect.

**Theorem.** *There is no projective plane of order $n = 1$.*

*Proof.* Assume there is a projective plane of order 1. Let $\ell$ be a line with exactly 2 points on it; such a line exists by Axiom P2. Call the points $A$ and $B$. We consider two cases depending on whether $A$ and $B$ are 2 of the 4 points we know exist from Axiom P1. First, suppose that they are. Then there are 2 other points $P$ and $Q$ not on $\ell$ such that no 3 of $A$, $B$, $P$, and $Q$ are collinear. By Axiom P3, there exists a line $\ell'$ passing through $P$ and $Q$. Since $A$, $B$, $P$, and $Q$ are not collinear, $\ell \neq \ell'$. By Axiom P4, there is a point $R$ on both $\ell$ and $\ell'$. Since no 3 of $A$, $B$, $P$, and $Q$ are collinear, we easily deduce that $R$ is not $A$ or $B$. But then $A$, $B$, and $R$ are 3 points on $\ell$, contrary to the fact that $\ell$ has only 2 points on it.

Now, suppose $A$ and $B$ are not 2 of the 4 points we know exist from Axiom P1. Then from Axiom P1, there exist 3 points, say $P$, $Q$, and $R$, not on $\ell$ which are noncollinear. Hence, by Axiom P3, there are 3 distinct lines $\ell_1$, $\ell_2$, and $\ell_3$ passing through $P$ and $Q$, $P$ and $R$, and $Q$ and $R$, respectively. Since $P$, $Q$, and $R$ are not on $\ell$, none of these lines can be $\ell$. By Axiom P4, each of these lines intersects $\ell$. Axiom P3 implies that these intersection points must be unique. Hence, we get a contradiction again to the fact that $\ell$ has only 2 points on it. $\square$

**Theorem (Dual of Axiom P1).** *There exist at least 4 distinct lines, no 3 of which are concurrent.*

*Proof.* By Axiom P1, there exist 4 distinct points no 3 of which are collinear. Call them $A$, $B$, $C$, and $D$. By Axiom P3, there exist lines $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ passing through $A$ and $B$, $B$ and $C$, $C$ and $D$, and $A$ and $D$, respectively. Since no 3 of $A$, $B$, $C$, and $D$ are collinear, each of $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ passes through exactly 2 of the points $A$, $B$, $C$, and $D$ and the lines $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ are distinct. We will complete the proof by showing that no 3 of these 4 lines are concurrent.

Assume 3 (or more) of the lines $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ intersect at a common point $P$. Then the above implies that $P \neq A$, $P \neq B$, $P \neq C$, and $P \neq D$. Observe that given any 3 of the 4 lines $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$, from among those 3 lines, there must be at least 2 which have one of $A$, $B$, $C$, or $D$ in common. Thus, by considering the 3 (or more) lines among $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ which intersect at $P$, we can find at least 2 lines which intersect at $P$ and at some other point ($A$, $B$, $C$, or $D$). This contradicts Axiom P3, so our assumption that $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ intersect at a common point $P$ must be incorrect. Therefore, no 3 of the 4 lines $\ell_1$, $\ell_2$, $\ell_3$, and $\ell_4$ are concurrent. $\square$