<u>Math 532/736I: Modern Geometry</u>

Spring 2016

Test 2: Solution Key

1) a) P = (1008,0) **b)** Q = (1008,1008) **c)** T = (1008,-1008)

2) $f = R_{\frac{\pi}{2},(0,2016)} R_{\frac{\pi}{2},(1,1)}$

$$R_{\frac{\pi}{2},(0,2016)} = \begin{bmatrix} 0 & -1 & 2016 \\ 1 & 0 & 2016 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{\frac{\pi}{2},(1,1)} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 2016 \\ 1 & 0 & 2016 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2016 \\ 0 & -1 & 2018 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2016 \\ 0 & -1 & 2018 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} -x + 2016 \\ -y + 2018 \\ 1 \end{bmatrix}$$

$x = -x + 2016$ $\qquad$ $y = -y + 2018$

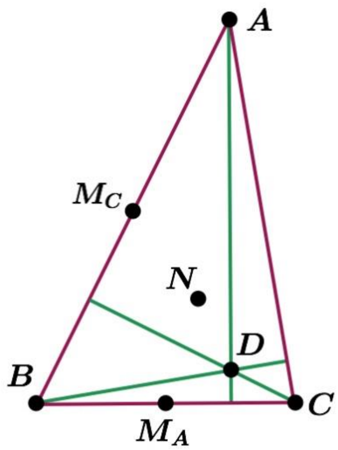
$2x = 2016$ $\qquad$ $2y = 2018$

$x = 1008$ $\qquad$ $y = 1009$

$f\ (1008,1009) = (1008,1009)$ $\qquad$ $\angle = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

$$\boxed{f = R_{\pi,(1008,1009)}}$$

3)



- $M_A = Midpoint\ of\ \overline{BC}$
- $M_C = Midpoint\ of\ \overline{AB}$
- $D = Intersection\ of\ altitudes\ of\ \Delta ABC$
- $N = \frac{A+B+C+D}{4}$

$$\left|\overline{NM_A}\right| = \left|\overline{NM_C}\right|$$
$$\Leftrightarrow (N-M_A)^2 = (N-M_C)^2$$
$$\Leftrightarrow (N-M_A)^2 - (N-M_C)^2 = 0$$
$$\Leftrightarrow (N-M_A+N-M_C)(N-M_A-N+M_c) = 0$$
$$\Leftrightarrow (N-M_A+N-M_C)(\cancel{N}-M_A-\cancel{N}+M_c) = 0$$
$$\Leftrightarrow (2N-M_A-M_c)(M_C-M_A) = 0$$
$$\Leftrightarrow (\tfrac{A+B+C+D}{2}-\tfrac{B+C}{2}-\tfrac{A+B}{2})(\tfrac{A+B}{2}-\tfrac{B+C}{2}) = 0$$
$$\Leftrightarrow (\tfrac{\cancel{A}+\cancel{B}+\cancel{C}+D}{2}-\tfrac{\cancel{B}+\cancel{C}}{2}-\tfrac{\cancel{A}+B}{2})(\tfrac{A+\cancel{B}}{2}-\tfrac{\cancel{B}+C}{2}) = 0$$
$$\Leftrightarrow \tfrac{1}{4}(D-B)(A-C) = 0$$

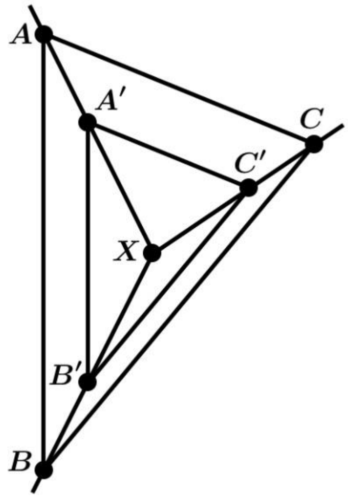
Recall:

$$X^2 = Y^2$$
$$X^2 - Y^2 = 0$$
$$(X+Y)(X-Y) = 0$$

In this case let:

$$X = (N-M_A)$$
$$Y = (N-M_c)$$

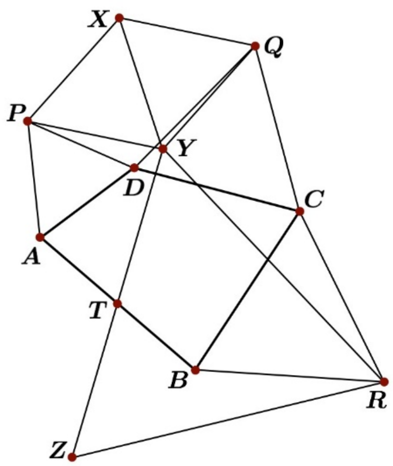
We know that $\overline{DB}$ is on the altitude to $\overline{AC}$ so they must be perpendicular and our result holds. Following the above logic backwards, we arrive at our intended conclusion.

4) 

- $\boldsymbol{A, B, C}$ ***are noncollinear as are*** $\boldsymbol{A', B', C'}$
- $\boldsymbol{A \neq A', B \neq B', \& \ C \neq C'}$
- $\boldsymbol{\overleftrightarrow{AA'}, \overleftrightarrow{BB'}, \& \ \overleftrightarrow{CC'}}$ ***intersect at X***
- $\boldsymbol{\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'} \ \& \ \overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}}$

- $k_1 = \boxed{k_2}$
- $\left(\frac{k_1}{k_1-k_2}\right)A + \left(-\frac{k_2}{k_1-k_2}\right)B = \boxed{\left(\frac{1-k_2}{k_1-k_2}\right)B' - \left(\frac{1-k_1}{k_1-k_2}\right)A'}$
- $t = \boxed{-\frac{k_2}{k_1-k_2}}$
- $t = \boxed{-\frac{1-k_1}{k_1-k_2}}$
- $k_2 = \boxed{k_3}$
- $k_3 = \boxed{k_1}$
- $k_1A - k_3C = \boxed{(1-k_3)C' - (1-k_1)A'}$
- $\boxed{k_1(A-C)} = (1-k_1)(C'-A')$
- vector $\boxed{\overrightarrow{AC}}$
- line $\boxed{\overleftrightarrow{AC}}$

5)



- $\Delta PAD, \Delta QDC, \Delta RCB, \Delta PXY, \Delta QXY$ & ΔRYZ **are all equilateral**
- $\boldsymbol{M\ is\ midpoint\ of\ \overline{AB}}$
- $\boldsymbol{\overline{AB}}$ & $\boldsymbol{\overline{YZ}\ intersect\ at\ T}$
- $\boldsymbol{f = R_{\pi,M} R_{\frac{\pi}{3},R} R_{\frac{\pi}{3},Q} R_{\frac{\pi}{3},P}}$

a) $f(A) = R_{\pi,M} R_{\frac{\pi}{3},R} R_{\frac{\pi}{3},Q} R_{\frac{\pi}{3},P}(A)$

$f(A) = R_{\pi,M} R_{\frac{\pi}{3},R} R_{\frac{\pi}{3},Q}(D)$

$f(A) = R_{\pi,M} R_{\frac{\pi}{3},R}(C)$

$f(A) = R_{\pi,M}(B)$

$f(A) = \boxed{A}$

b). Since $\pi + \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = 2\pi$, Theorem 2 implies that f is a translation.
From part (a) we found that $f(A) = A$ which implies that the translation is by (0,0) so we get that $f = \boxed{T_{(0,0)}}$

c) We know $f(Y) = Y$ since part (b) tells us $f = T_{(0,0)}$ and

$f(Y) = R_{\pi,M} R_{\frac{\pi}{3},R} R_{\frac{\pi}{3},Q} R_{\frac{\pi}{3},P}(Y)$

$f(Y) = R_{\pi,M} R_{\frac{\pi}{3},R} R_{\frac{\pi}{3},Q}(X)$

$f(Y) = R_{\pi,M} R_{\frac{\pi}{3},R}(Y)$

$f(Y) = R_{\pi,M}(Z)$

So $Y = R_{\pi,M}(Z)$. Hence, M is the midpoint of $\overline{YZ}$. Since M is defined in the problem as the midpoint of $\overline{AB}$, M is on both $\overline{AB}$ and $\overline{YZ}$. Therefore, M=T.