MATH 532, 736I: MODERN GEOMETRY

Test 1, Spring 2019 Name Solutions

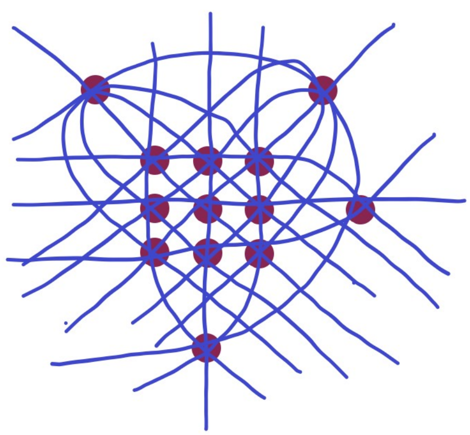
Show All Work

Instructions: This test consists of 3 pages of problems. Put your name at the top of this page and at the top of the first page of the packet of blank paper given to you. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Show ALL of your work. Do NOT use a calculator.

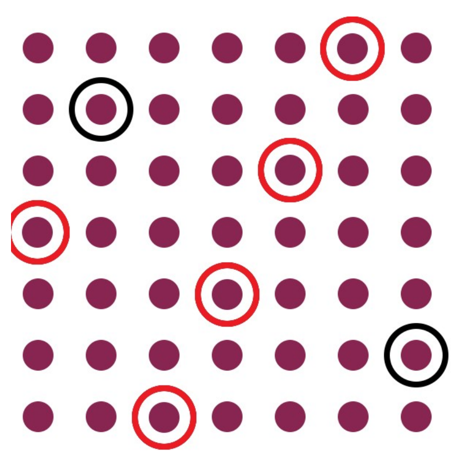
Points: Part I (50 pts), Part II (50 pts)

Part I. The point value for each problem appears to the left of each problem. In Problem 5, I will assume you are using the axioms as you state them in your answer to Problem 1 below.

10 pts (1) Draw a model for a finite PROJECTIVE plane of order 3. Be sure to clearly draw every point and every line in your model. If I look at your model, I should be able to tell where each of your points and lines are. In particular, make sure that each line you draw cannot be mistaken for two lines. (Note: you do not need to label your points and lines.)



10 pts (2) Two points have been circled in the 7×7 array of points to the right. Using the model for a finite affine plane of order 7 discussed in class, finish circling the points that belong to the same line as the given circled points. (You do not need to use the packet of blank paper for this problem.)



$\boxed{\text{12 pts}}$ (3) Consider the points $(4, 14)$ and $(20, 3)$ in a 31×31 array of points for our model of a finite affine plane of order 31. Find the equation of the line passing through these two points. Put your answer below in the form $y \equiv mx + k \pmod{31}$ where m and k are among the numbers $0, 1, 2, \ldots, 30$. Be sure to show your work in the packet provided with this test. You will not get credit for a correct answer without correct work.

Answer: $\boxed{y \equiv 9x + 9 \pmod{31}}$ (Make sure m and k are in $\{0, 1, 2, \ldots, 30\}$.)

Given the two points, we can know that the slope of the desired line would be $(3-14)/(20-4) = -11/16$, so the line we're looking for would be of the form $y = (-11/16)x + k$, except that we want to work modulo 31. To calculate the slope modulo 31, we observe that

$$\begin{aligned} m \equiv \frac{-11}{16} \pmod{31} &\implies 16m \equiv -11 \pmod{31} \\ &\implies m \equiv -22 \pmod{31} \\ &\implies m \equiv 9 \pmod{31}, \end{aligned}$$

where the second implication follows from multiplying both sides of the previous congruence by 2. This gives us $y \equiv 9x + k \pmod{31}$ for some k. So, now, we can plug in a point to solve for the intercept. We use the point $(4, 14)$ to see that

$$\begin{aligned} 14 \equiv 9(4) + k \pmod{31} &\implies 14 \equiv 36 + k \pmod{31} \\ &\implies k \equiv -22 \pmod{31} \\ &\implies k \equiv 9 \pmod{31}. \end{aligned}$$

This gives us a final answer of $y \equiv 9x + 9 \pmod{31}$.

$\boxed{\text{18 pts}}$ (4) Recall that the axioms for a finite projective plane are

Axiom P1: There exist at least 4 points no 3 of which are collinear.
Axiom P2: There exists at least 1 line with exactly $n+1$ (distinct) points on it.
Axiom P3: Given 2 distinct points, there is exactly 1 line that they both lie on.
Axiom P4: Given 2 distinct lines, there is at least 1 point on both of them.

Fill in the boxes below to complete a proof of the result stated below. Note that this proof has some different wording than from previous tests, and you need to make the argument given here correct and so cannot rely completely on what was in blanks on the prior tests for the course.

> Result: *If ℓ is a line with exactly $n+1$ points on it in a finite projective plane of order n and A is a point not on ℓ, then there exist at least $n+1$ distinct lines passing through A.*

This is part of a result done in class and that you were to have learned for this test. Use only the axioms stated above to complete the proof.

Proof. Let $P_1, P_2, \ldots, P_{n+1}$ be the $n+1$ distinct points on ℓ. Since

$\boxed{A \text{ is not on } \ell\text{, and each } P_j \text{ for } j \in \{1, 2, \ldots, n+1\} \text{ is on } \ell}$

we have that $A \neq P_j$ for each $j \in \{1, 2, \ldots, n+1\}$. By $\boxed{\text{Axiom P3}}$, there is a line ℓ_j passing through A and P_j for each $j \in \{1, 2, \ldots, n+1\}$. Since $\boxed{A \text{ is not on } \ell}$ and A is on ℓ_j, we see that $\ell_j \neq \ell$ for each $j \in \{1, 2, \ldots, n+1\}$. We justify next that

$\boxed{\ell_i \neq \ell_j \text{ for } j \text{ and } i \text{ in } \{1, 2, \ldots, n+1\} \text{ with } j \neq i}$.

Assume $\boxed{\ell_i = \ell_j}$ for some i and j in $\{1, 2, \ldots, n+1\}$ with $i \neq j$. Then the two points $\boxed{P_i \text{ and } P_j}$ are both on ℓ_i. Since these two points are distinct, $\boxed{\text{Axiom P3}}$ implies that there is $\boxed{\text{exactly one line}}$ passing through them. Since $\boxed{P_i \text{ and } P_j}$ are two points that are both on ℓ_i and are both on ℓ, we deduce $\boxed{\ell_i = \ell}$. This contradicts that $\boxed{\ell_i \neq \ell}$. Thus, our assumption is wrong and the lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ are different. This finishes the proof that there are at least $n+1$ distinct lines passing through A. ■

Part II. The point values appear to the left of each problem. The problems in this section all deal with an axiomatic system consisting of the following axioms.

Axiom 1. There exists at least one line.

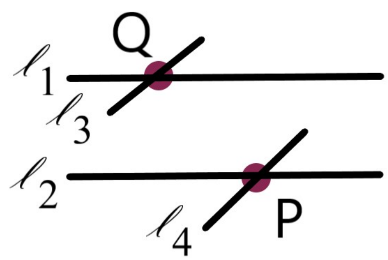
Axiom 2. Given any line, there is another line which is parallel to it.

Axiom 3. Given any line, there is another line which intersects it.

8 pts

(1) Justify that the axiomatic system is consistent.

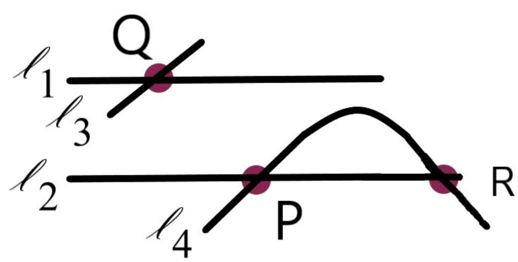
Here, we see a model - two lines ℓ_1 and ℓ_3 which intersect at point Q, and two other lines, ℓ_2 and ℓ_4 which intersect at point P, but neither ℓ_1 nor ℓ_3 intersect with ℓ_2 or ℓ_4, which satisfies the axioms.



6 pts

(2) Justify that the axiomatic system is *not* complete. Include some brief explanation, in complete English sentences, for your answer.

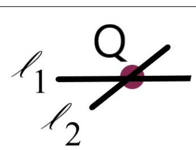
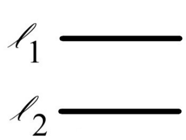
Here, we see a second model which fulfills the axioms, but is substantively different from the model in (1) above: to wit, ℓ_2 and ℓ_4 intersect at both point P and point R.



12 pts

(3) Justify that the axiomatic system is independent.

For each axiom, we give a model below which does not satisfy that axiom but does follow the other two axioms. Thus, each axiom is independent of the other two, and the axiomatic system is independent.

Axiom 1:	$\varnothing$
Axiom 2:	
Axiom 3:	

8 pts (4) Write the dual of Axiom 2 below. (Write in clear English. Do not refer to parallel points.)

Dual of Axiom 2. Given any point, there is another point which is not collinear with the given point.

16 pts (5) Prove that there are at least 4 lines in any model for this axiomatic system.

By Axiom 1, there is a line ℓ_1. By Axiom 3, there must be another line that intersects ℓ_1; we'll call it ℓ_2. By Axiom 2, there must also be a line parallel to ℓ_1; we'll call it ℓ_3. Since ℓ_3 is parallel to ℓ_1 and ℓ_2 intersects ℓ_1, these three lines are distinct.

To find a fourth line, first, we consider the possibility that ℓ_2 and ℓ_3 are parallel. In this case, we use Axiom 3 to see that there must be a line that intersects ℓ_3 and call it ℓ_4. Since ℓ_3 is parallel to both ℓ_1 and ℓ_2 in this case and ℓ_4 is not, we see that the four lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are distinct.

Next, we consider the possibility that ℓ_2 and ℓ_3 intersect. Then ℓ_2 intersects both ℓ_1 and ℓ_3. By Axiom 2, there must be a line ℓ_4 parallel to ℓ_2. Since each of ℓ_1 and ℓ_3 is not parallel to ℓ_2, we see that in this case there are at least four distinct lines, namely the lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4.

Thus, in any case, we deduce that there are at least four lines for any model in this axiomatic system, completing the proof.