MATH 532, 736I: MODERN GEOMETRY

Test 1 Solutions

Test 1 (2011):

Part I:

(1) Axioms for a finite AFFINE plane of order *n*

Axiom A1: There exist at least 4 distinct points no 3 of which are collinear.

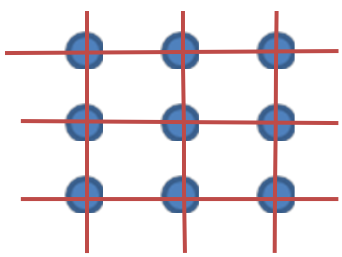
Axiom A2: There exists at least 1 line with exactly n points on it.

Axiom A3: Given any 2 distinct points, there exists exactly one line passing through the 2 points.

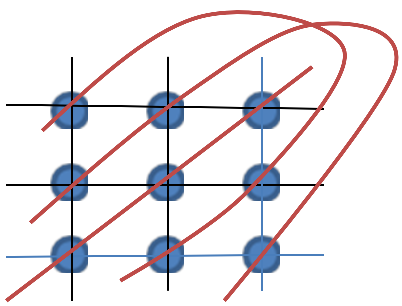
Axiom A4: Given any line l and any point P not on l, there is exactly 1 line through P that does not intersect l.

(2) Give a model for a finite PROJECTIVE plane of order 3

a) a. Illustrates lines with slope o and undefined

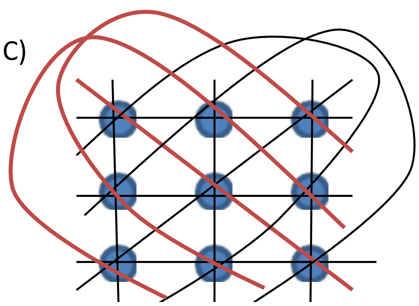


b)



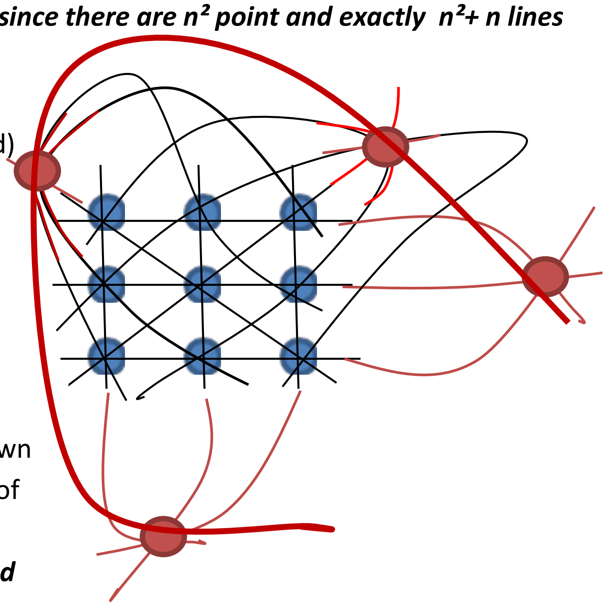
b. Illustrates lines with slope 1

C)



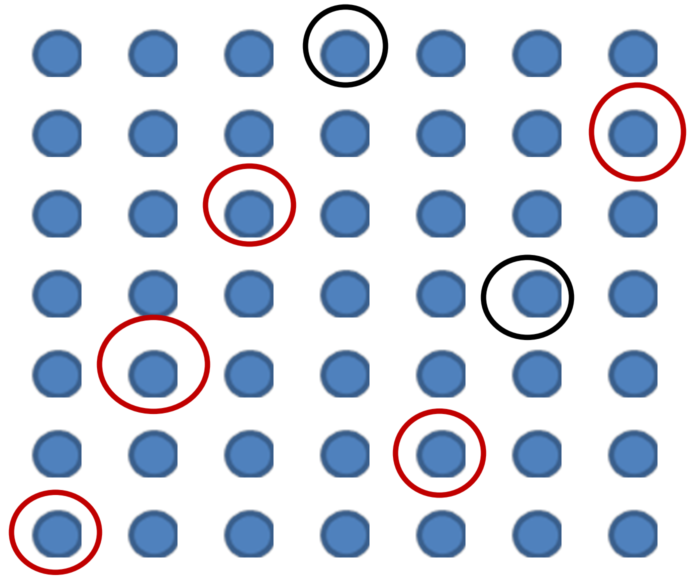
c. Illustrates lines with slope -1. ***Notice, at this point we have drawn a model for an Affine plane of order 3, since there are n^2 point and exactly $n^2 + n$ lines***

d)



d. Illustrates points at infinity and a line connecting them. We have drawn a model of a finite Projective plane of order 3. ***Notice that there are $n^2 + n + 1$ points, $n^2 + n + 1$ lines, and $n + 1$ points on each line.***

(3) Two points have been circled in the 7x7 array of points to the right. Using the model for a finite affine plane of order 7 discussed in class, finish circling the points that belong to the same line as the given circled points.



(4) Consider the points (4, 2) and (10, 9) in an 11x11 array of points for our model of a finite affine plane of order 11. Find the equation of the line passing through these two points. Put your answer in the form $y \equiv mx + k \pmod{11}$

a. Find the slope of the line by using the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

You only need to use one point to find the *k*. However, for information sake, both are shown. By the slope formula given above, we find m= 7/6. Note that 7 = 18 (mod 11) so we can use m = 18/6 = 3 for the slope.

a) Using the point (4, 2)

$y \equiv 3x + k \pmod{11}$

$2 \equiv 3(4) + k \pmod{11}$

$2 \equiv 12 + k \pmod{11}$

$2 - 12 \equiv k \pmod{11}$

$-10 \equiv k \pmod{11}$

b) Using the point (10, 9)

$y \equiv 3x + k \pmod{11}$

$9 \equiv 3(10) + k \pmod{11}$

$9 \equiv 30 + k \pmod{11}$

$9 - 30 \equiv k \pmod{11}$

$-21 \equiv k \pmod{11}$

Definition: if $a \equiv b \pmod{m}$, the m divides (a-b).

So we know that, $-21 \equiv k \pmod{11} \Longrightarrow 11\ divides\ (21 + k) \Longrightarrow k = 1$,

since k is in {0,1,…,10}

(5) Using only the theorem below and the axioms you stated in Problem 1, fill in the boxes below to complete a proof that in an affine plane of order n, for each line ℓ, there are at least n-1 lines parallel to ℓ. This is part of a proof you were to have memorized for class. (More precisely, the problem you were to have memorized for class involved showing that there are exactly n-1 such lines. I have not used the word "exactly" in my statement of the problem above.)

Theorem: *In an affine plane of order n, each line contains exactly n point.*

Note: The theorem is to be used in the proof below. The proof is establishing that there are at least n-1 lines parallel to ℓ as stated above.

Proof: Let ℓ be an arbitrary line. By $\boxed{\text{Axiom } A_1,}$ there is a point P_1 not on ℓ. $\boxed{\text{By the given Theorem,}}$ line ℓ has at least one point, say P_2, on it. From Axiom A_3, there is a line ℓ' passing through P_1 and P_2. Since $\boxed{P_1 \text{ is on } \ell' \text{ and not on } \ell,}$ we have that $\ell' \neq \ell$. $\boxed{\text{Axiom } A_3}$ implies that P_2 is the only point on both $\ell' and\ \ell$. $\boxed{\text{By the given Theorem,}}$ ℓ' has exactly $\boxed{n}$ points on it, two of which are P_1 and P_2. Let $P_3, \ldots, P_n$ denote the remaining points on. ℓ' For each $j \neq 2$, P_j is not on ℓ so that $\boxed{\text{Axiom } A_4}$ implies that there is a line ℓ_j passing through P_j and parallel to ℓ. Each such ℓ_j is different from ℓ' since $\boxed{\ell_j \text{ is parallel to } \ell \text{ and } \ell' \text{ is not.}}$ It follows that the lines ℓ_j (with $j \neq 2$) are distinct by $\boxed{\text{Axiom } A_3}$ (since ℓ' is the unique line passing through any two of the $P_j's$). Thus there are at least n-1 distinct line parallel to ℓ (namely, the lines ℓ_j with $j \neq 2$). ■

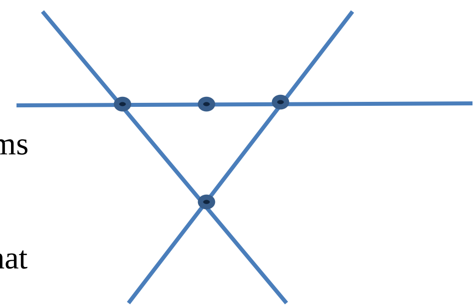
Part II: The problems in this section all deal with an axiomatic system consisting of the following axioms.

Axiom 1. There exist 3 collinear points (that is, 3 points and a line with the 3point on the line).
Axiom 2. There exist exactly 3 distinct lines.
Axiom 3. Given two distinct lines, there is at least one point on both lines.
<u>Axiom 4</u>. Given two distinct points, there is at most one line passing through them.

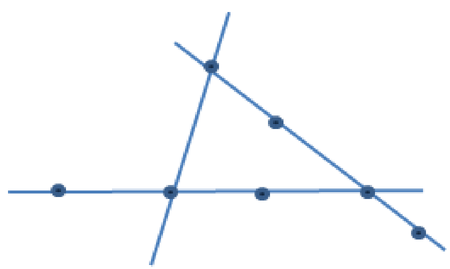
Note: Axiom 1 is saying that there exist 3 collinear points. This does NOT mean “exactly”. There may be more points in the axiomatic system, and there may even be more point on the same line as these 3 points.

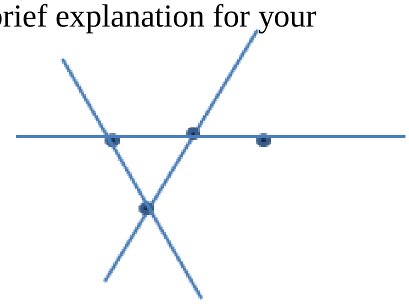
1) Justify that the axiomatic system is consistent.

The model satisfies all the axioms in the system, therefore the axiomatic system is consistent
NOTE: this is just one model that satisfies the axioms.



2) Justify that the axiomatic system in not compete. Include some brief explanation for your answer.

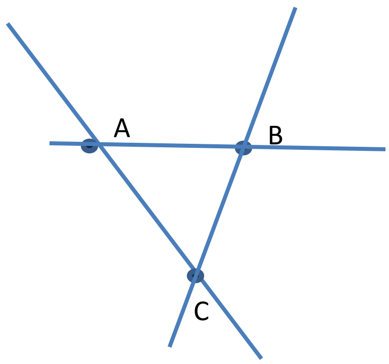




The axiomatic system is not complete because there exist more than 1 model for the system, and it is possible for a theorem about the system to hold for one model but not for another. Ie: in the model on the left there exist 7 points which is not true for the model on the right.

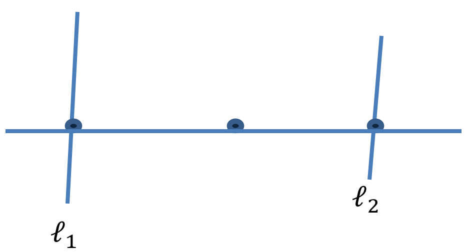
NOTE: Because we can add at least one point to at least one line when constructing our model , implies that we can have as many points as desired. Although we need only show two, there are infinitely many models.

3) Justify that the axiomatic system is independent.

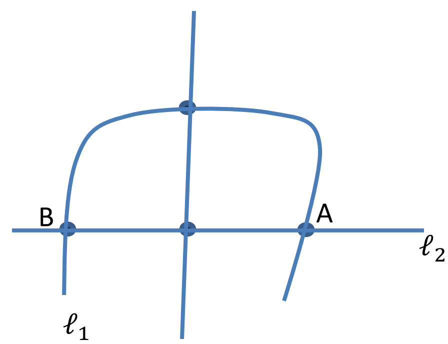


Axioms 2-4 hold, Axiom 1 does not. (A, B, C are noncolinear; therefore, Axiom 1 is independent.)

Axioms 1,3,and 4 hold; axiom 2 does not (ℓ is the only line that exists; therefore, Axiom 2 is independent).



Axioms 1, 2, and4 hold; Axiom 3 does not (no point on both ℓ_1 andℓ_2; therefore, Axiom 3 is independent).



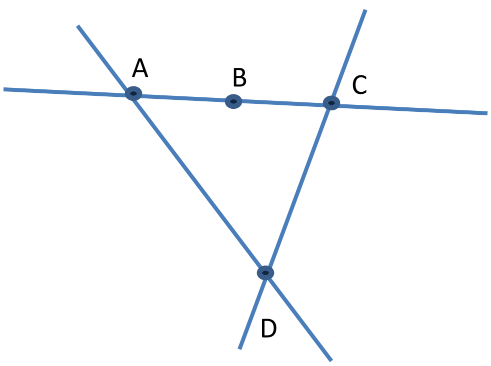
Axioms 1-3 hold; Axiom 4 does not (ℓ_1 and ℓ_2 both pass through A and B; therefore, Axiom 4 is independent).

4) What is the dual of Axiom 3?

Dual Axiom 3: *Given 2 distinct points, there is at least one line passing through both points.*

5) Justify that the principle of duality does not hold for this axiomatic system.

Because the system is not complete, we can use any model for the system and show that the dual of one of the axioms does not hold for that particular model and hence, the principle of duality does not hole for the entire system.



Using the information from question 4, the Dual of Axiom 3*: Given 2 distinct points, there is at least one line passing through both points. (*IN the model to the left, there is no line passing through points B and D). Hence the dual for Axiom 3 does not hold and the principle of duality for the axiomatic system does not hold.

OR

Using the Dual of Axiom 2: *There exist exactly 3 distinct points* (In the model to the left, there exist 4 distinct points, A, B, C, and D). Hence the dual for Axiom 2 does not hold and the principle of duality for the axiomatic system does not hold

6) Prove the following:

Given two distinct line, there is exactly one point on both lines.

Let ℓ and ℓ' be two arbitrary lines. By Axiom 3, there is at least 1 point $\mathcal{P}$ on both ℓ and ℓ'. Assume that there is a point $\mathcal{Q} \neq \mathcal{P}$ also on ℓ and ℓ'. Since $\mathcal{P}$ and $\mathcal{Q}$ are 2 distinct points with 2 lines, ℓ and ℓ', passing through them, we get a contradiction to Axiom 4. Therefore, $\mathcal{P}$ is the only point on both ℓ and ℓ' and two lines intersect in exactly 1 point.■