# MATH 532/736I, LECTURE NOTES 9

**The Nine-Point Circle Theorem.** *Let $A$, $B$ and $C$ be the three vertices of a triangle. Let $M_A$ be the midpoint of $\overline{BC}$, $M_B$ be the midpoint of $\overline{AC}$, and $M_C$ be the midpoint of $\overline{AB}$. Let $\overline{AP_A}$ be an altitude for $\triangle ABC$ (so $P_A$ is on $\overline{BC}$). Let $\overline{BP_B}$ be an altitude for $\triangle ABC$ (so $P_B$ is on $\overline{AC}$). Let $\overline{CP_C}$ be an altitude for $\triangle ABC$ (so $P_C$ is on $\overline{AB}$). Let $D$ be the intersection of these three altitudes. Let $Q_A$ be the midpoint of $\overline{AD}$, $Q_B$ be the midpoint of $\overline{BD}$, and $Q_C$ be the midpoint of $\overline{CD}$. Then the nine points $M_A$, $M_B$, $M_C$, $P_A$, $P_B$, $P_C$, $Q_A$, $Q_B$ and $Q_C$ all lie on a circle, and the center of this circle is $(A+B+C+D)/4$.*

**Lemma.** *Let $A$, $B$ and $P$ be three distinct points. Let $O = (A+B)/2$. If $\angle APB$ is a right angle, then $OP = OA$ $(= OB)$.*

**Comment:** This is a well-known geometric fact; the point $P$ is on the circle centered at $O$ of radius $OA$. But this can also be shown with vectors. To see this, use that the following three equations are equivalent:

$$\left(\frac{A+B}{2} - P\right)^2 = \left(\frac{A+B}{2} - A\right)^2$$

$$(A+B-2P)^2 = (B-A)^2$$

$$(B-P)(A-P) = 0.$$

The last equation follows from the line above it by using that $X^2 - Y^2 = (X+Y)(X-Y)$. Also, note that the last equation above holds since $\angle APB$ is a right angle.

**Basic Ideas of Proof of The Nine-Point Circle Theorem:**

- Note $Q_A = (A+D)/2$, $Q_B = (B+D)/2$ and $Q_C = (C+D)/2$.
- Let $N = (A+B+C+D)/4$.
- Observe that $N$ is the midpoint of $M_A$ and $Q_A$, $N$ is the midpoint of $M_B$ and $Q_B$, and $N$ is the midpoint of $M_C$ and $Q_C$. So $NM_A = NQ_A$, $NM_B = NQ_B$ and $NM_C = NQ_C$.
- A computation shows $(N - M_A)^2 = (N - M_B)^2 = (N - M_C)^2$. For example, begin with the first equation and rewrite it as

  $$(A-B-C+D)^2 = (-A+B-C+D)^2.$$

  Show that this follows from $\overrightarrow{AB}$ and $\overrightarrow{CD}$ being perpendicular.
- The above implies $NM_A$, $NQ_A$, $NM_B$, $NQ_B$, $NM_C$ and $NQ_C$ are all equal.
- Using the lemma and that $\triangle Q_AP_AM_A$ is a right triangle, deduce that $NP_A = NQ_A = NM_A$. Similarly $\triangle Q_BP_BM_B$ and $\triangle Q_CP_CM_C$ are right triangles, and the proof of the theorem follows.