MATH 532, 736I: MODERN GEOMETRY

Some Solutions to Old Final Exam Problems

Final Exam (1992):

Part II:

(2) (a) Let a, b, x, and y be such that $B = (a, b)$ and $A = (x, y)$. Then

$$\begin{aligned} T_B R_{\alpha,A} &= T_{(a,b)} R_{\alpha,(x,y)} \\ &= \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & x(1-\cos(\alpha)) + y\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) & -x\sin(\alpha) + y(1-\cos(\alpha)) \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & x(1-\cos(\alpha)) + y\sin(\alpha) + a \\ \sin(\alpha) & \cos(\alpha) & -x\sin(\alpha) + y(1-\cos(\alpha)) + b \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Also, $A + B = (x + a, y + b)$. Thus,

$$\begin{aligned} R_{\alpha,A+B} T_B &= R_{\alpha,(x+a,y+b)} T_{(a,b)} \\ &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & (x+a)(1-\cos(\alpha)) + (y+b)\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) & -(x+a)\sin(\alpha) + (y+b)(1-\cos(\alpha)) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & u \\ \sin(\alpha) & \cos(\alpha) & v \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where

$$u = a\cos(\alpha) - b\sin(\alpha) + (x+a)(1-\cos(\alpha)) + (y+b)\sin(\alpha) = x(1-\cos(\alpha)) + y\sin(\alpha) + a$$

and

$$v = a\sin(\alpha) + b\cos(\alpha) - (x+a)\sin(\alpha) + (y+b)(1-\cos(\alpha)) = -x\sin(\alpha) + y(1-\cos(\alpha)) + b.$$

It follows that $T_B R_{\alpha,A} = R_{\alpha,A+B} T_B$.

(b) Note that in addition to (a) we have $T_A T_B = T_{A+B}$ and $R_{\alpha,A} R_{\beta,A} = R_{\alpha+\beta,A}$. Hence,

$$\begin{aligned} &R_{\pi/5,(1,2)} T_{(1,1)} R_{2\pi/5,(0,1)} T_{(-1,3)} R_{4\pi/5,(1,-2)} T_{(2,-1)} R_{3\pi/5,(-1,-1)} \\ &\qquad = R_{\pi/5,(1,2)} R_{2\pi/5,(1,2)} T_{(1,1)} T_{(-1,3)} R_{4\pi/5,(1,-2)} T_{(2,-1)} R_{3\pi/5,(-1,-1)} \\ &\qquad = R_{\pi/5,(1,2)} R_{2\pi/5,(1,2)} T_{(0,4)} R_{4\pi/5,(1,-2)} T_{(2,-1)} R_{3\pi/5,(-1,-1)} \\ &\qquad = R_{\pi/5,(1,2)} R_{2\pi/5,(1,2)} R_{4\pi/5,(1,2)} T_{(0,4)} T_{(2,-1)} R_{3\pi/5,(-1,-1)} \end{aligned}$$

$$= R_{\pi/5,(1,2)} R_{2\pi/5,(1,2)} R_{4\pi/5,(1,2)} T_{(2,3)} R_{3\pi/5,(-1,-1)}$$
$$= R_{\pi/5,(1,2)} R_{2\pi/5,(1,2)} R_{4\pi/5,(1,2)} R_{3\pi/5,(1,2)} T_{(2,3)}$$
$$= R_{\frac{\pi}{5}+\frac{2\pi}{5}+\frac{4\pi}{5}+\frac{3\pi}{5},(1,2)} T_{(2,3)}$$
$$= R_{2\pi,(1,2)} T_{(2,3)} = T_{(2,3)}.$$

Therefore, the result indicated in the problem follows with $C = (2, 3)$.

Part III:

(1) Yes, a projective plane of order 2 will serve as a model satisfying the given axioms.

(2) No. Axiom 1 implies there are 3 points, say P, Q, and R, that are not collinear. By Axiom 3, there is a line ℓ passing through Q and R. Since P, Q, and R are not collinear, P is a point not on ℓ. Thus, Axiom 5 follows from Axioms 1 and 2 (Axiom 5 is not independent of the other axioms).

(3) No. Besides the model given in (1), a projective plane of order 3 serves as a model satisfying the given axioms. Since a projective plane of order 2 and a projective plane of order 3 have a different number of points (the first has 7 points and the second has 13 points), these two models are not isomorphic. Hence, the axiomatic system is not complete.

(4) Yes. [An explanation is omitted. Using the axioms, you can show that the dual of each axiom holds.]

Part IV:

(2) We are given that

$$A_n = R_{\pi,P_n} R_{\pi,P_{n-1}} \cdots R_{\pi,P_2} R_{\pi,P_1}(A) \qquad \text{and} \qquad B_n = R_{\pi,P_n} R_{\pi,P_{n-1}} \cdots R_{\pi,P_2} R_{\pi,P_1}(B).$$

By considering the sum of the angles in the rotations, we obtain from Theorem 1 (or the more general result done in class) that

$$R_{\pi,P_n} R_{\pi,P_{n-1}} \cdots R_{\pi,P_2} R_{\pi,P_1} = \begin{cases} R_{\pi,C} & \text{if } n \text{ is odd} \\ T_D & \text{if } n \text{ is even,} \end{cases}$$

where C and D represent points (depending on n and the points P_j).

If n is even, then $A_n = R_{\pi,C}(A)$ and $B_n = R_{\pi,C}(B)$. If A, B, and C are collinear, then it follows that A, B, A_n, and B_n are all on a line. If A, B, and C are not collinear, then the triangles ΔACB and ΔA_nCB_n are congruent (by side-angle-side). In this case, $\angle ABC = \angle A_nB_nC$ so that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A_nB_n}$ are parallel. Also, $\overline{AB}$ and $\overline{A_nB_n}$ have the same length. Thus, A, B, A_n, and B_n are the vertices of a parallelogram.

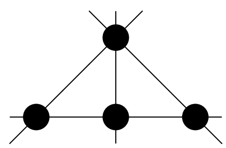
If n is odd, then $A_n = T_D(A)$ and $B_n = T_D(B)$. It follows that $\overrightarrow{AA_n} = \overrightarrow{BB_n}$ so that $\overleftrightarrow{AA_n}$ and $\overleftrightarrow{BB_n}$ are either parallel or coincide. Also, $\overline{AA_n}$ and $\overline{BB_n}$ have the same length. Thus, A, B, A_n, and B_n are either on a line or they are the vertices of a parallelogram.

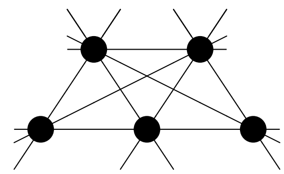
Final Exam (1993):

Part I:

(4) Axiom 3 does not hold. The lines $y \equiv 2x \pmod 4$ and $y \equiv 0 \pmod 4$ both pass through the points $(0,0)$ and $(2,0)$. Also, Axiom 4 does not hold. For $P = (0,0)$ and ℓ the line $y \equiv 1 \pmod 4$, both of the lines $y \equiv 2x \pmod 4$ and $y \equiv 0 \pmod 4$ pass through P and do not intersect ℓ.

(5) (a) Yes. The following models satisfy the axioms.





(b) Axiom 1 is not independent from the other axioms. By Axiom 3, there is a line ℓ with exactly 2 distinct points on it. By Axiom 4, there is a line ℓ' with exactly 3 distinct points on it. Clearly, $\ell \neq \ell'$ (there are a different number of points on them). Observe that no two of the 3 points on ℓ' can be on ℓ since otherwise Axiom 2 implies $\ell = \ell'$ which is not the case. It follows that there is at least one point, say A, on ℓ that is not on ℓ' and there are at least two points, say B and C, on ℓ' not on ℓ. By Axiom 2, ℓ' is the only line passing through B and C. Since ℓ' does not pass through A, the points A, B, and C are noncollinear. Hence, Axiom 1 follows from Axioms 2, 3, and 4.

(c) No, see the answer to (a). The models given there are not isomorphic since one contains 4 points and the other contains 5 points.

(d) Assume otherwise so that every two lines intersect. By Axiom 3, there is a line ℓ with exactly 2 distinct points on it. By Axiom 4, there is a line ℓ' with exactly 3 distinct points on it. Clearly, $\ell \neq \ell'$. Since ℓ and ℓ' intersect, there is some point D on both lines. Let A be on ℓ and B and C be on ℓ' with A, B, and C different from D. By Axiom 2, A is not on ℓ', and both B and C are not on ℓ. By Axiom 5, there is a fifth point E not equal to A, B, C, or D. Observe that E is not on ℓ and E is not on ℓ'. By Axiom 2, there is a line ℓ_1 passing through E and B and there is a line ℓ_2 passing through E and C. Note that since E is on each of these lines and not on ℓ', we have $\ell_1 \neq \ell'$ and $\ell_2 \neq \ell'$. We deduce that $\ell_1 \neq \ell_2$ since otherwise ℓ_1 is a line passing through B and C and, hence, by Axiom 2 we would have $\ell_1 = \ell'$ (which is not the case). Neither ℓ_1 nor ℓ_2 can pass through D since otherwise by Axiom 2 we would have that one of ℓ_1 and ℓ_2 equals ℓ' (which is not the case). Also, since ℓ_1 and ℓ_2 both pass through E and $\ell_1 \neq \ell_2$, we deduce from Axiom 2 that at least one of ℓ_1 and ℓ_2 does not pass through A. It follows that at least one of ℓ_1 and ℓ_2 does not pass through both A and D; in other words, at least one of ℓ_1 and ℓ_2 does not intersect ℓ. This proves there exist 2 lines which do not intersect.

Part II:

(3) (a) By Theorem 3, $R_{\pi/2,Y}R_{\pi/2,X} = R_{\pi,C}$ for some point C. By Theorem 3 again, $R_{\pi,M}R_{\pi,C} = T_P$ for some point P. Thus,

$$f = R_{\pi,M}R_{\pi/2,Y}R_{\pi/2,X} = R_{\pi,M}R_{\pi,C} = T_P.$$

Since $R_{\pi/2,X}(B) = A$ and $R_{\pi/2,Y}(A) = C$ and $R_{\pi,M}(C) = B$, we deduce that $f(B) = B$. Since f is a translation that takes B to B, it follows that f is the identity translation.

(b) Since f is the identity translation, $f(X) = X$. Clearly, $X = R_{\pi/2,X}(X)$. Let $X' = R_{\pi/2,Y}(X)$. Then $f(X) = X$ implies $R_{\pi,M}(X') = X$. Since $X' = R_{\pi/2,Y}(X)$, we have $XY = X'Y$. Since $X = R_{\pi,M}(X')$, we deduce that $XM = X'M$. Since YM is equal to itself, we obtain that ΔXYM is congruent to $\Delta X'YM$. Thus, $\angle XYM = \angle X'YM$. Since $X' = R_{\pi/2,Y}(X)$, we have

$$2\angle X'YM = \angle XYM + \angle X'YM = \angle XYX' = \frac{\pi}{2}.$$

Also, $\angle YMX = \angle YMX'$ and $X = R_{\pi,M}(X')$ imply

$$2\angle YMX' = \angle YMX + \angle YMX' = \angle XMX' = \pi.$$

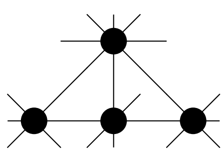
We obtain $\angle XYM = \angle X'YM = \pi/4$, $\angle YMX = \angle YMX' = \pi/2$, and consequently $\angle YXM = \angle YX'M = \pi/4$. It follows that ΔXMY is an isosceles right triangle.

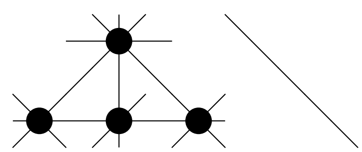
(4) This is not a complete answer, but the basic idea of this problem is to first explain why f equals $R_{(2\pi/3),T}$ (the main part of the problem) and then to give some explanation from there as to why $f(B) = I$.

Final Exam (1994):

(3) To see where the lines intersect, we solve for x in the congruence $20x + 17 \equiv 23x + 19 \pmod{41}$. In other words, we want to solve $3x \equiv -2 \pmod{41}$. One checks that $x \equiv 13 \pmod{41}$ is the solution. Since $20 \cdot 13 + 17 \equiv 277 \equiv 31 \pmod{41}$, we deduce $a = 13$ and $b = 31$.

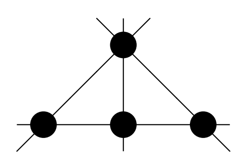
(5) (a) The axiomatic system is consistent since either of the following models satisfy all the axioms.





Note that for each model above and for each point in the model, there is a line passing through the point that does not pass through any other point. Also, the model on the right simply contains one more line than the model on the left.

(b) A single line with exactly 3 points on it satisfies all the axioms but Axiom 1. A single line with exactly 2 points on it satisfies all the axioms but Axiom 2. Two parallel lines, one passing through exactly 2 points and one passing through exactly 3 points, satisfies all the axioms but Axiom 3. The model to the right satisfies all the axioms but Axiom 4.



(c) The two models given in part (a) satisfy the axioms of the axiomatic system but they contain a different number of lines. Hence, there are at least two non-isomorphic models for this system.

(d) No. The dual of Axiom 1 is that there is at least one point with exactly 2 lines passing through it. In the models in part (a), each point has more than 2 lines passing through it. This shows that the dual of Axiom 1 is not true for some model for the axiomatic system. Hence, the principal of duality does not hold for the axiomatic system.

(e) By Axiom 1, there is a line m_1 passing through exactly 2 points. By Axiom 2, there is a line m_2 passing through exactly 3 points. We take $\ell = m_1$ if m_1 and m_2 intersect. If m_1 and m_2 do not intersect, we take ℓ to be a line passing through a point on m_1 and a point on m_2 (such a line exists by Axiom 3; ℓ can be any such line). In either case, $\ell \neq m_2$ and ℓ and m_2 intersect. By Axiom 3, at least 2 of the 3 points on m_2 are not on ℓ. Call these two points A and B. By Axiom 4, there is a line ℓ_1 parallel to ℓ passing through A. By Axiom 4, there is a line ℓ_2 parallel to ℓ passing through B. Since ℓ and m_2 intersect, the lines ℓ_1 and ℓ_2, being parallel to ℓ, cannot equal m_2. Axiom 3 then implies that ℓ_1 and ℓ_2 are distinct (otherwise, $\ell_1 = \ell_2$ and m_2 are distinct lines passing through A and B). Thus, ℓ is a line with at least 2 distinct lines, ℓ_1 and ℓ_2, parallel to it.

(6) Since $(\pi/2)+\pi+(\pi/2) = 2\pi$, f is a translation. If we begin with the point $(0,-1)$ and rotate about $(0,0)$ by $\pi/2$, the point is rotated to the point $(1,0)$. If we then rotate this point about $(1,0)$ by π, the point stays at $(1,0)$. Finally, if we take this point and rotate it about the point $(0,1)$ by $\pi/2$, the result is the point $(1,2)$. Thus, $f(0,-1) = (1,2)$. Since f is a translation, we deduce $f = T_{(1,3)}$.

(7) Triangles $\Delta PBB'$ and $\Delta RCC'$ are perspective from point Q.

(9) Since f is a rotation, $f = R_{\theta,(a,b)}$ for some angle θ and real numbers a and b. Since $f(0,52) = (0,25)$, the center point (a,b) of the rotation must be on the perpendicular bisector of the points $(0,52)$ and $(0,25)$ (as all points equidistant from these are on the perpendicular bisector). It follows that $b = (25+52)/2 = 77/2$. Similarly, $f(39,0) = (x,0)$, implies that (a,b) is on the perpendicular bisector of $(39,0)$ and $(x,0)$ so that $a = (x+39)/2$. Recalling that $f(0,52) = (0,25)$, we see that the triangle with vertices (a,b), $(0,77/2)$, and $(0,25)$ has interior angles $\theta/2$, $\pi/2$, and $(\pi-\theta)/2$. Similarly, since $f(39,0) = (x,0)$, the triangle with vertices (a,b), $((x+39)/2,0)$, and $(39,0)$ has interior angles $\theta/2$, $\pi/2$, and $(\pi-\theta)/2$. Thus, these two triangles are similar. Hence, the corresponding ratios of their sides must be equal. In particular,

$$\frac{x+39}{27} = \frac{(x+39)/2}{27/2} = \frac{77/2}{(x-39)/2} = \frac{77}{x-39}.$$

Thus, $x^2 - 39^2 = 27 \cdot 77$ so that $x^2 = 39^2 + 27 \cdot 77 = 3600$. Therefore, $x = 60$ or $x = -60$ (both are possible).

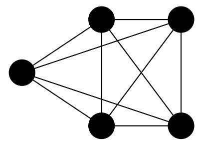
(10) See Part IV (2) of the 1992 final.

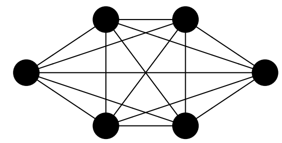
(11) The 3 lines $\ell = \overleftrightarrow{CD}$, $\overleftrightarrow{UM}$, and $\overleftrightarrow{VN}$ are parallel. Let X be the point at infinity corresponding to their common slope. Then each of these lines passes through X. The 2 triangles ΔABC and ΔVUX in the extended Euclidean plane are perspective from the point D (convince yourself of this). It follows that these 2 triangles are perspective from a line. This in turn implies K, M, and N are collinear.

Final Exam (1995):

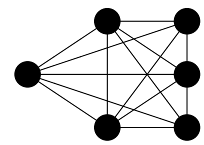
(3) The slope m satisfies $-6m \equiv 5 \pmod{17}$. Since $-6 \cdot 2 \equiv 5 \pmod{17}$, we deduce $m \equiv 2 \pmod{17}$. Thus, $y \equiv 2x + k \pmod{17}$. Since $(14,6)$ is a point on the array, $6 \equiv 2 \cdot 14 + k \pmod{17}$ so that $k \equiv 6 - 2 \cdot 14 \equiv -22 \equiv 12 \pmod{17}$. Thus, $m = 2$ and $k = 12$.

(5) (a) The axiomatic system is consistent since either of the following models satisfy all the axioms.





(b) A single line with two points on it would satisfy all the axioms except Axiom 1. The model to the right shows that there is a model for which every axiom but Axiom 2 holds. For a model satisfying all the axioms but Axiom 3, one can use two lines and two points with each line passing through both points. A triangle (with three points and three lines each passing through two points) satisfies every axiom but Axiom 4.



(c) The two models given in part (a) satisfy the axioms of the axiomatic system but they contain a different number of points. Hence, there are at least two non-isomorphic models for this system.

(d) No. The dual of Axiom 2 is that each point has exactly two lines passing through it. In the first model in part (a), each point has exactly four lines passing through it (or use the second model and note that each point has exactly five lines passing through it). This shows that the dual of Axiom 2 is not true for this model. Hence, the principal of duality does not hold for the axiomatic system.

(e) By Axiom 1, there exist at least 2 distinct lines. Call them ℓ and ℓ'. By Axiom 2, ℓ passes through exactly two points, say A and B. By Axiom 3, at most one of A and B can be on ℓ'. By Axiom 2, there must be at least one point on ℓ' different from A and B. Call it C. Note that C is not on ℓ. In particular, $C \neq A$ and $C \neq B$. By Axiom 4, there are at least two distinct lines ℓ_1 and ℓ_2 passing through C and parallel to ℓ. By Axiom 2, there is a point $D \neq C$ on ℓ_1 and there is a point $E \neq C$ on ℓ_2. Note that $D \neq E$ by Axiom 3. Also, each of D and E is different from both A and B since D and E are on lines parallel to ℓ which passes through A and B. Thus, A, B, C, D, and E are distinct points. We are done if we can show that no three of these five points are collinear. But, in fact, Axiom 2 implies that any three distinct points cannot be collinear. Hence, the desired conclusion follows.

(7) Triangles $\Delta XA'A$ and $\Delta YB'B$ are perspective from point Z.

(9) Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$ are perpendicular and $\overleftrightarrow{CB}$ and $\overleftrightarrow{CD}$ are perpendicular, we deduce

$$\begin{aligned} 0 &= (B-C)(D-C) - (B-A)(D-A) = -CD - CB + C^2 + AD + AB - A^2 \\ &= \left(C^2 - A^2\right) - CB + AB - CD + AD = (C-A)(C+A-B-D). \end{aligned}$$

Since X is the midpoint of $\overline{AC}$ and Y is the midpoint of $\overline{BD}$, we obtain $X = (A+C)/2$ and $Y = (B+D)/2$. Hence, $\overrightarrow{YX} = X - Y = (A+C-B-D)/2$. Since $\overrightarrow{AC} = C - A$, we deduce from $(C-A)(C+A-B-D) = 0$ that $\overleftrightarrow{XY}$ and $\overleftrightarrow{AC}$ are perpendicular.

(10) Observe that ΔABG and ΔCDF are perspective from the point E. By Desargues' Theorem, these triangles must also be perspective from a line. The lines $\overleftrightarrow{BG}$ and $\overleftrightarrow{DF}$, being parallel, intersect in the *extended* Euclidean plane at a point at infinity. The lines $\overleftrightarrow{AG}$ and $\overleftrightarrow{CF}$, being parallel, intersect in the *extended* Euclidean plane at a point at infinity. It follows that ΔABG and ΔCDF are perspective from the line at infinity. This implies that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point at infinity. In other words, in the (unextended) Euclidean plane, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.